Chapter 5

Queue with ignored interruption in random environment and self correction.

Introduction

In the previous chapter we discussed queueing models with interruption in Markovian environment with partially ignored interruption. In this chapter we analyze two queueing models. In the first model we consider a single server queueing system with arrival following Poisson process and service time Erlang distributed. At times there is a possibility for interruption in service process. Interruption occurs according to a Poisson process and
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Each interruption duration is exponentially distributed. Assume the interruption does not affect the customer in service. Further at a time at most one interruption affects any service. The service continues ignoring the interruption. During interrupted service there is a scope for self correction of interruption. Self correction occurs according to Poisson process. On the onset of interruption an interruption clock is started which is Erlang distributed. If the interruption clock is realized before service completion the server goes for repair and after repair the service is resumed. Repair time is exponentially distributed. If service is completed before the realization of interruption clock the next customer in the head of the queue enters for service.

In the second model the arrival process and the service process are as in the first model. During service interruption may occur. Interruption to service occurs according to a Poisson process. There are $n$ environmental factors causing interruption. Interruption due to $i^{th}$ environmental factor occurs with probability $p_i$. If the interruption is due to first $m$ factors it is ignored and service continues. But the service will be at lower rate. The duration up to which the server works without breakdown is assessed with the help of an interruption clock. This clock starts ticking with the initiation of the first interruption to the service of a customer. The duration of the clock is exponentially distributed. During that period there is a possibility for self correction of interruption. This self correction duration is exponentially distributed. If self correction occurs the service rate changes. On realization of the interruption clock the server goes for repair. The repair time is exponentially distributed. After repair the interrupted service is resumed. If the service of a customer is completed with interruption the next customer in the head of the queue enter for
service in the interrupted server. If the interruption is due to the remaining $n - m$ factors the server directly goes for repair. Taking into account the severity of interruption caused by these $n - m$ factors, protection for remaining service is provided at the epoch of resumption of service after repair. The stability of both the systems are analyzed. Steady state probability vector is calculated using matrix analytic method. Important performance measures are numerically substantiated.

5.1 Model Description (Model I)

Consider a single server queueing system in which arrival occurs according to a Poisson process with parameter $\lambda$. The service time is Erlang distributed with shape and scale parameters $\mu$ and $a$, respectively. During service there is a possibility for occurrence of interruption to service. The duration of interruption is exponentially distributed with parameter $\beta$. The service is continued without attending the interruption. A clock called interruption clock, is started on the onset of interruption. The interruption clock is Erlang distributed with shape and scale parameters $\delta$ and $b$ respectively. Sometimes the interruption gets self corrected. The self correction occurs according to a Poisson process with parameter $\gamma$. If the customer in service completes service with interruption the next customer in the head of the queue enters for service. If the interruption clock realizes before completion of service of the customer, the server goes for repair and after repair the service to the interrupted customer is resumed. The repair time is exponentially distributed with parameter $\eta$. The model is pictorially represented in Figure 5.1.
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5.2 Analysis of the model

The Markov process associated with the queueing model, $X = \{X(t), t \geq 0\}$ is a continuous time Markov Chain, where $X(t) = (N(t), S(t), I_1(t), I_2(t))$. Here at time $t$:

- $N(t)$ - Number of customers in the system;
- $I_1(t)$ - Phase of interruption clock. It varies from 1 to $b$;
- $I_2(t)$ - Phase of service. It varies from 1 to $a$;
- $S(t)$ - Status of server;

$$S(t) = \begin{cases} 
0, & \text{if a service is going on without interruption;} \\
1, & \text{if service is going on with interruption;} \\
2, & \text{if server is under repair.}
\end{cases}$$

The state space associated with $X$ is $\{0\} \cup \{n, 0, j\} \cup \{n, 1, i, j\} \cup \{n, 2, j\}; n = 1, 2, \ldots; i = 1, \ldots, b; j = 1, \ldots, a$. The infinitesimal generator matrix, $Q$ as-
5.2. Analysis of the model

A queueing model is associated with the queueing model:

\[
Q = \begin{bmatrix}
B_0 & B_1 \\
B_2 & A_1 & A_0 \\
& A_2 & A_1 & A_0 \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
\end{bmatrix}
\]

where \( B_0 = [-\lambda], \) \( B_1 = \begin{bmatrix} \lambda & 0 \end{bmatrix}_{1 \times a(2+b)} \).

\( B_2 \) is a column matrix of order \((2+b)a \times 1\).

\[
B_2(i, 1) = \begin{cases} 
\mu, & \text{for } i = ra, r = 1, 2, \ldots, b + 1; \\
0, & \text{otherwise}; 
\end{cases}
\]

\( A_2, A_1, \) and \( A_0 \) are square matrices of order \( a(b+2) \).

\[
A_0 = \lambda I_{a(b+2)}.
\]

\[
A_2(i, j) = \begin{cases} 
\mu, & \text{for } i = ra, j = (r - 1)a + 1, r = 1, 2, \ldots, b + 1; \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
A_1 = \begin{bmatrix}
C_0 & C_1 & 0 \\
C_2 & C_3 & C_4 \\
C_5 & 0 & C_6
\end{bmatrix}
\]

where

\[
C_0(i, j) = \begin{cases} 
-\mu - \lambda - \beta, & \text{for } i = j = 1, \ldots, a; \\
\mu, & \text{for } j = i + 1, i = 1, \ldots, a - 1; \ i, j = 1, \ldots, a \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
C_1(i, j) = \begin{cases} 
\beta, & \text{for } i = j; i = 1, \ldots, a; \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
C_2 = e_b \otimes \gamma I_a.
\]

\( C_3 \) is a square matrix of order \( ab \).

\[
C_3(i, j) = \begin{cases} 
\omega, & \text{if } i = j; \\
\delta, & \text{for } j = i + a; i = 1, \ldots, (b - 1)a; \\
\mu, & \text{for } i = (r - 1)a + l; j = i + 1, r = 1, \ldots, b, l = 1, \ldots, a - 1; \\
\mu, & \text{for } i = ra; j = (r - 1)a + 1, r = 1, \ldots, b; 
\end{cases}
\]

where \( \omega = -\mu - \lambda - \beta - \delta. \)
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5.2.1 Steady-state analysis

Let $\pi$ denote the steady-state probability vector of the generator $A_0 + A_1 + A_2$. That is, $\pi(A_0 + A_1 + A_2) = 0$; $\pi e = 1$: The LIQBD description of the model indicates that the queueing system is stable if and only if $\pi A_0 e < \pi A_2 e$. The vector, $\pi$, cannot be obtained explicitly in terms of the Here $\pi A_0 e = \lambda$, $\pi A_2 e = \left( \sum_{i=a}^{(b+1)a} \pi_i \right) \mu$

and the condition for stability is $\lambda < \left( \sum_{i=a}^{(b+1)a} \pi_i \right) \mu$,

where $\sum_{i=a}^{(b+1)a} \pi_i = \frac{1}{a} \left( 1 - \frac{1}{\frac{\eta}{\beta} (1 + \frac{\gamma}{\delta})^b + \frac{\eta}{\beta} \sum_{r=1}^{b-1} (1 + \frac{\gamma}{\delta})^{b-r} + (1 + \frac{\eta}{\delta})} \right)$.

5.2.2 Stationary distribution

Let $\chi= (x_0, x_1, x_2, ...)$ be the steady state probability vector of the Markov chain $\{X(t), t \geq 0\}$. Each $x_i = (x_{i0}, x_{i1}, x_{i2})$, $i > 0$. $x_{i0}$ is a vector with $a$ elements, $x_{i1}$ is a vector with $ab$ elements and $x_{i2}$ is a vector with $a$ elements. We assume that $x_2 = x_1.R$, and $x_i = x_1.R^{i-1}, i \geq 2$, where $R$ is the minimal non-negative solution to the matrix quadratic equation
5.3. Performance measures

\[ R^2 A_2 + RA_1 + A_0 = 0. \]
From \( \chi Q = 0 \) we get
\[ x_0 B_0 + x_1 B_2 = 0. \]
\[ x_0 B_1 + x_1 (A_1 + RA_2) = 0. \]
Solving the above two equations we get \( x_0 \) and \( x_1 \) subject to the normalizing condition \( x_0 e + x_1 (I - R)^{-1} e = 1. \)

5.3 Performance measures

In this section we list a number of key system performance measures to bring out the qualitative aspects of the model under study. These are listed below along with their formula for computation.

5.3.1 Expected Service Rate

Let \( Y = \{ Y(t), t \geq 0 \} \), where \( Y(t) = (S(t), I_1(t), I_2(t)) \) is a continuous time Markov chain representing the service process with interruption with \( a(b + 2) \) transient states and one absorbing state. The state space corresponding to \( Y \) is \( \{ 0, j \} \cup \{ 1, i, j \} \cup \{ 2, j \} \cup * \) where * is the absorbing state, \( i = 1, 2, \ldots, b; j = 1, 2, \ldots, a \). The infinitesimal generator associated with the service process \( \overline{Q} = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix} \) where
\[
S = \begin{bmatrix} C_0 + \lambda & C_1 & 0 \\ C_2 & C_3 + \lambda & C_4 \\ C_5 & 0 & C_6 + \lambda \end{bmatrix}
\]
The important results obtained from the analysis of the service process
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are

- The time until absorption $E(S) = -\theta S^{-1}e$ where $\theta = (1, 0, \ldots, 0)$.
- The time spent in each of the $a(b + 2)$ phases is given by $\theta S^{-1}$.
- The expected service rate $\mu_s = \frac{1}{E(S)}$.

5.3.2 Expected waiting time

The waiting time of the particular customer who joined as the $m^{th}$ customer in the queue is the time until absorption of the Markov Chain $W = \{W(t), t \geq 0\}$ where $W(t) = (\mathcal{N}(t), S(t), I_1(t), I_2(t))$. $\mathcal{N}(t)$ is the rank of the tagged customer and all other random variables are as defined in section 5.2. The state space corresponding to $W$ is $\{r, 0, j\} \cup \{r, 1, i, j\} \cup \{r, 2, j\} \cup \Omega$ where $\Omega$ is the absorbing state, $i = 1, 2, \ldots, b; j = 1, 2, \ldots, a; r = m, m - 1, \ldots, 1$. The infinitesimal generator associated with the waiting time is $Q' = \begin{bmatrix} W & W^0 \\ 0 & 0 \end{bmatrix}$

The waiting time of a tagged customer follows a phase type representation $(\sigma, W)$ where $W = \begin{bmatrix} S & S^0 \theta \\ S & S^0 \theta \\ \vdots & \vdots \\ S & S^0 \theta \end{bmatrix}$ and $W^0 = \begin{bmatrix} 0 \\ \vdots \\ S^0 \end{bmatrix}$

$\sigma$ is the initial probability vector which indicates that the chain is starting from level $r$.

- The expected waiting time of the $r^{th}$ customer is
5.3. Performance measures

\[ E_w = -S^{-1}(I - (S^0 \theta S^{-1})(r^{-1}))(I - S^0 \theta S^{-1})^{-1} e. \]

- The expected waiting time of general customer is \( E_w = \sum_{r=1}^{\infty} x_r E_{w_r}. \)

### 5.3.3 Expected number of interruptions during the service of a single customer

For calculating the expected number of interruptions we consider the Markov chain \( Z = \{Z(t), t \geq 0\} \) where \( Z(t) = (\hat{N}(t), S(t), I_1(t), I_2(t)) \). \( \hat{N}(t) \) is the number of interruptions occurred until time \( t \) and all other random variables are as defined in section 5.2. The state space corresponding to \( Z \) is \((0, 0, j) \cup (r, 1, i, j) \cup (r, 2, j) \cup (r, 0, j) \cup \nabla\) where \( \nabla\) represents the absorbing state, \( i = 1, 2, \ldots, b; j = 1, 2, \ldots, a; r = 1, 2, \ldots \). The infinitesimal generator associated with the Markov chain is \( \tilde{Q} = \begin{bmatrix} 0 & U_0 \\ 0 & 0 \end{bmatrix} \)

where \( U = \begin{bmatrix} B_0 & B_1 \\ \overline{A}_1 & \overline{A}_0 \\ \overline{A}_1 & \overline{A}_0 \\ \vdots & \vdots & \vdots \end{bmatrix} \) and \( U_0 = \begin{bmatrix} \overline{A}_2 \\ \overline{B}_2 \\ \overline{B}_2 \\ \vdots \end{bmatrix} \).

\( B_0 \) is a matrix of dimension \( a \).

\[ B_0(i, j) = \begin{cases} -\mu - \beta, & \text{for } i = j = 1, \ldots, a; \\ \mu, & \text{for } j = i + 1, i = 1, \ldots, a - 1; \\ 0 & \text{otherwise}; \end{cases} \]
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\[ \mathbf{B}_1 = \begin{bmatrix} \beta I_a & 0 \end{bmatrix}_{a \times a(b+2)}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 \\ \mu \end{bmatrix}_{a \times 1}, \]

\[ \mathbf{B}_2(i, 1) = \begin{cases} \mu, & \text{for } i = ra, i = a(b+2), r = 1, \ldots, b; \\ 0, & \text{otherwise}; \end{cases} \]

\[ \mathbf{A}_1 \text{ and } \mathbf{A}_0 \text{ are of dimension } a(b+2) \]

\[ \mathbf{A}_1(i, j) = \begin{cases} -\mu - \gamma - \delta, & \text{for } i = j = 1, \ldots, ab \\ -\eta, & \text{for } j = i = 1 + ab, \ldots, ab + a \\ -\mu - \beta, & \text{for } j = i = ab + a + 1, \ldots, a(b+2) \\ \delta, & \text{for } j = i + 1, i = 1, \ldots, ab; j = a + 1, \ldots, a(b + 1) \\ \mu, & \text{for } j = i + 1, i = (r-1)a + l, l = 1, \ldots, a - 1 \\ 0, & \text{otherwise}; \end{cases} \]

\[ \mathbf{A}_0(i, j) = \begin{cases} \beta, & \text{for } i = a + ab + j, j = 1, \ldots, a; \\ 0, & \text{otherwise}. \end{cases} \]

\( Z(t) \) follows a phase type distribution with representation \((\vartheta, U)\) where \( \vartheta \) is the initial probability vector.

Let \( y_j \) be the probability that there are exactly \( j \) interruptions during the service of a customer. Then

\[ y_j = \begin{cases} \vartheta (\mathbf{B}_0^{-1}\mathbf{A}_2), & \text{for } j = 0; \\ \vartheta (\mathbf{B}_0^{-1}\mathbf{B}_1) (\mathbf{A}_1^{-1}\mathbf{A}_0)^{j-1} (\mathbf{A}_1^{-1}\mathbf{B}_2) & \text{otherwise}. \end{cases} \]

### 5.3.4 Other important performance measures

- Probability that the system is idle \( P_{idle} = x_0 \).

- Probability that there are \( i \) customers in the system \( P_i = x_i e \)
• Expected number of customers in the system $E(C) = \sum_{i=1}^{\infty} ix_ie$.

• Probability for self correction of interruption before interruption clock realization $P_{\text{selfcorrection}} = \sum_{n=0}^{b-1} \frac{\gamma^n}{(\gamma + \delta)^{n+1}}$.

• Probability for service completion without any interruption $P(s) = \left[\frac{\mu^n}{(\mu + \beta)^n}\right]$.

• Probability that the system is working with interruption $P_{\text{Int}} = \sum_{i=1}^{\infty} x_{i1}e$

• Effective self correction rate $E_{\text{selfcorr}} = \eta \sum_{i=1}^{\infty} x_{i1}e$

• Probability that the server is under repair $P_{\text{rep}} = \sum_{i=1}^{\infty} x_{i2}e$

• Effective interruption rate $E_{\text{Int}} = \beta \sum_{i=1}^{\infty} x_{i0}e$

• Effective repair rate $E_{\text{rep}} = \eta \sum_{i=1}^{\infty} x_{i1}e$

5.4 Numerical illustrations

The results obtained in the previous sections are numerically illustrated in this section. Assume arbitrary values for parameters $\mu = 5$, $\gamma = .2$, $\delta = 3$, $\eta = 5$ and $\beta = 3$. 


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Effect of $\lambda$ on various performance measures

Table 5.1: Effect of $\lambda$ on various performance measures

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$E(C)$</th>
<th>$P_{idle}$</th>
<th>$E_{selfcorr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>0.2275</td>
<td>0.8056</td>
<td>0.0150</td>
</tr>
<tr>
<td>1</td>
<td>0.4761</td>
<td>0.6540</td>
<td>0.0277</td>
</tr>
<tr>
<td>1.5</td>
<td>0.7441</td>
<td>0.5388</td>
<td>0.0380</td>
</tr>
<tr>
<td>2</td>
<td>1.0280</td>
<td>0.4519</td>
<td>0.0460</td>
</tr>
<tr>
<td>2.5</td>
<td>1.3243</td>
<td>0.3860</td>
<td>0.0523</td>
</tr>
<tr>
<td>3</td>
<td>1.6299</td>
<td>0.3353</td>
<td>0.0572</td>
</tr>
<tr>
<td>3.5</td>
<td>1.9424</td>
<td>0.2956</td>
<td>0.0611</td>
</tr>
<tr>
<td>4</td>
<td>2.2600</td>
<td>0.2639</td>
<td>0.0643</td>
</tr>
</tbody>
</table>

From Table 5.1 as the value of $\lambda$ increases expected number of customers in the system $E(C)$ and expected self correction rate begins to increase but probability for idleness decreases which are on expected lines.

Effect of $\mu$ on various performance measures

Taking $\lambda = 2$ and all other values as above we get the following values for different performance measures on the variation in $\mu$. As service rate $\mu$

Table 5.2: Effect of $\mu$ on various performance measures

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$E(S)$</th>
<th>$E(C)$</th>
<th>$P_{idle}$</th>
<th>$P(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>.5670</td>
<td>1.2243</td>
<td>0.4125</td>
<td>0.1250</td>
</tr>
<tr>
<td>4</td>
<td>0.4935</td>
<td>1.1237</td>
<td>0.4310</td>
<td>0.1866</td>
</tr>
<tr>
<td>5</td>
<td>0.4341</td>
<td>1.0280</td>
<td>0.4519</td>
<td>0.2441</td>
</tr>
<tr>
<td>6</td>
<td>0.3861</td>
<td>0.9394</td>
<td>0.4743</td>
<td>0.2963</td>
</tr>
<tr>
<td>7</td>
<td>0.3467</td>
<td>0.8588</td>
<td>0.4974</td>
<td>0.3430</td>
</tr>
<tr>
<td>8</td>
<td>0.3141</td>
<td>0.7862</td>
<td>0.5206</td>
<td>0.3847</td>
</tr>
</tbody>
</table>
increases expected service time $E(S)$ and expected number of customers in the system decreases $E(C)$, but probability for idleness and probability for service completion without any interruption increases (refer Table 5.2).

### Effect of $\eta$ on various performance measures

Assuming $\mu = 4$ we get the following values for different performance measures corresponding to the variation in $\eta$:

As the repair rate increases expected service time $E(S)$ and expected number of customers in the system $E(C)$ decrease, but probability for idleness and effective interruption rate increase which are on expected lines (Table 5.3).

#### Table 5.3: Effect of $\eta$ on various performance measures

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$E(S)$</th>
<th>$E(C)$</th>
<th>$P_{Idle}$</th>
<th>$E_{Int}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5397</td>
<td>1.5271</td>
<td>0.3986</td>
<td>0.2839</td>
</tr>
<tr>
<td>2</td>
<td>0.5108</td>
<td>1.2491</td>
<td>0.4180</td>
<td>0.5943</td>
</tr>
<tr>
<td>3</td>
<td>0.5012</td>
<td>1.1754</td>
<td>0.4251</td>
<td>0.9054</td>
</tr>
<tr>
<td>4</td>
<td>0.4964</td>
<td>1.1423</td>
<td>0.4287</td>
<td>1.2166</td>
</tr>
<tr>
<td>5</td>
<td>0.4935</td>
<td>1.1237</td>
<td>0.4310</td>
<td>1.5278</td>
</tr>
<tr>
<td>6</td>
<td>0.4916</td>
<td>1.1118</td>
<td>0.4325</td>
<td>1.8391</td>
</tr>
</tbody>
</table>

### Effect of $\delta$ on various performance measures

From Table 5.4, as the realization rate of interruption clock increases expected service time $E(S)$, expected number of customers in the system $E(C)$ and effective interruption rate $E_{Int}$ increase, but probability for
idleness and effective self correction rate $E_{selfcorr}$ decrease which are on expected lines. When interruption clock realization rate increases rate of repair increases which causes the increase in effective service time of a customer. This leads to the increase in number of customers in the system. As a result the probability for idleness decreases.

### 5.5 Model description (Model II)

We consider a single server queueing model in which customers arrive according to a Poisson process with rate $\lambda$. Service time is Erlang distributed with shape and scale parameters $\mu$ and $a$ respectively. During service there is a chance for interruption. There are $n$ environmental factors causing interruption to service. These $n$ factors are numbered 1 to $n$ depending on the ascending order of severity of interruption caused by these factors. Interruption to service occurs according to Poisson process with parameter $\beta$. Interruption due to $i^{th}$ environmental factor occurs with probability $p_i$. If the interruption is due to first $m(m < n)$ factors it is ignored and service is continued. But the service is at a lower rate $\mu_i, i = 1, 2, \ldots, m$. This interruption clock is started simultaneously with the occurrence of interruption. It is exponentially distributed with
5.5. Model description (Model II)

parameter $\delta_i, i = 1, 2 = \ldots, m$. During interrupted service period there is a possibility for self correction of interruption. This self correction is exponentially distributed with parameter $\gamma_i, i = 1, 2, \ldots, m$. If self correction occurs the service rate changes from $\mu_i$ to $\mu$. On realization of the interruption clock the server goes for repair. The repair time is exponentially distributed with parameter $\eta_i, i = 1, 2, \ldots, m$. After repair the interrupted service is resumed. If the service of a customer is completed with interruption the next customer in the head of the queue enter for service with the server in interruption.

If the interruption is due to the remaining $n - m$ factors the server directly goes for repair. Taking into account the severity of interruption caused by these $n - m$ factors, protection for remaining service is provided at the epoch of resumption of service after repair.

Figure 5.2: Model description
5.6 Mathematical description

The queueing model described above can be mathematically formulated as a Markov chain. Let \( X = \{X(t), t \geq 0\} = \{(N(t), S(t), I_1(t), I_2(t)), t \geq 0\} \) where \( N(t) \) is the number of customers in the system, \( S(t) \) is the status of the server, \( I_1(t) \) is the environmental factor causing interruption and \( I_2(t) \) is the phase of service:

\[
S(t) = \begin{cases} 
0, & \text{when a service facing so far no interruption} \\
1, & \text{if interrupted service going on} \\
2, & \text{if server under repair} \\
3, & \text{if protected service is going on.}
\end{cases}
\]

The state space of the process is

\[
\{(r, 0, i) \cup (r, 1, j, i) \cup (r, 2, k, i) \cup (r, 3, i); r = 1, \ldots, \infty; i = 1, \ldots, a; j = 1, \ldots, m; k = 1, \ldots, n; k = 1, \ldots, n\} \cup \nabla \text{ where } \nabla \text{ represents the absorbing state. Absorbing state means the customer moving out from the system after service completion. The infinitesimal generator matrix of the process is given by}
\]

\[
Q = \begin{bmatrix}
B_0 & B_1 \\
B_2 & A_1 & A_0 \\
 & A_2 & A_1 & A_0 \\
 & & \ddots & \ddots & \ddots \\
 & & & \ddots & \ddots \\
 & & & & \ddots & \ddots
\end{bmatrix}
\]

where \( B_0 = [-\lambda], \quad B_1 = \begin{bmatrix} \lambda & 0 \end{bmatrix} \)

\( B_2 \) is a column matrix of order \((2 + m + n)a \times 1\) and

\[
B_2(i, 1) = \begin{cases} 
\mu, & \text{for } i = a, \& i = (2 + m + n)a; \\
\mu_j, & \text{for } i = a(r + 1); r = 1, 2, \ldots, m; \\
0, & \text{otherwise.}
\end{cases}
\]
5.6. Mathematical description

\[ A_0 = \lambda I_{(2+m+n)a} \quad A_1 = \begin{bmatrix} C_0 & C_1 & C_2 & 0 \\ C_3 & C_4 & C_5 & 0 \\ C_6 & 0 & C_7 & C_8 \\ 0 & 0 & 0 & C_9 \end{bmatrix}^{(2+m+n)a \times (2+m+n)a} \]

\( C_0 \) is a matrix of order \( a \),

\[ C_0(i, j) = \begin{cases} -\lambda - \beta - \mu, & \text{for } i = j; \\ \mu, & \text{for } j = i + 1; i = 1, \ldots, a - 1 \\ 0, & \text{otherwise}. \end{cases} \]

\( C_1 = (\beta p' \otimes I_a)_{a \times am} \), \( C_2 = \begin{bmatrix} 0 & \beta p'' \otimes I_a \end{bmatrix}_{a \times am} \) where \( p = (p', p'') \) with \( p' = (p_1, \ldots, p_m) \) and \( p'' = (p_{m+1}, \ldots, p_n) \).

Let \( \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix} \) then \( C_3 = \gamma \otimes I_a \) and \( C_4 \) is a matrix of order \( ma \),

\[ C_4(i, j) = \begin{cases} \theta_r, & \text{for } i = j; i = (r - 1)a + l; r = 1, \ldots, m; l = 1, \ldots, a; \\ \mu_r, & \text{for } i + 1 = j; i = (r - 1)a + l; r = 1, \ldots, m; l = 1, \ldots, a - 1; \\ 0, & \text{otherwise}. \end{cases} \]

where \( \theta_r = -\lambda - \mu_r - \gamma_r - \delta_r \).

\( C_5 \) is a matrix of order \( ma \times na \),

\[ C_5(i, j) = \begin{cases} \delta_t, & \text{for } i = j; i = (t - 1)a + k; t = 1, \ldots, m; k = 1, \ldots, a \\ 0, & \text{otherwise}; \end{cases} \]

Let \( \eta = \begin{bmatrix} \eta' \\ \eta'' \end{bmatrix} \) with \( \eta' = (\eta_1, \eta_2, \ldots, \eta_m)^T \) and \( \eta'' = (\eta_{m+1}, \ldots, \eta_n)^T \)

then \( C_6 = \begin{bmatrix} \eta' \otimes I_a \\ 0 \end{bmatrix}_{(na \times a)} \).

\( C_7 \) is a matrix of order \( na \),
Chapter 5. Queue with ignored interruption in random environment and self correction.

\[ C_7 = \begin{cases} -\lambda - \eta_r, & \text{for } i = j; i = (r-1)a + k, r = 1, \ldots, n; k = 1, \ldots, a \\ 0, & \text{otherwise.} \end{cases} \]

\[ C_8 = \begin{bmatrix} \frac{0}{\eta''} \otimes I_a \end{bmatrix}_{(na \times a)}. \]

\[ C_9 \] is a matrix of order \( a \),

\[ C_9 = \begin{cases} -\lambda - \mu, & \text{for } i = j; i = (r-1)a + k, k = 1, \ldots, a; \\ \mu, & \text{for } j = i + 1; i = (r-1)a + k, k = 1, \ldots, a - 1; \\ 0, & \text{otherwise.} \end{cases} \]

\[ A_2(i, j) = \begin{cases} \mu, & \text{for } i = a; j = 1; \text{and } i = (m + n + 2)a j = 1; \\ \mu_r, & \text{for } i = (r+1)a; j = ra + 1, r = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases} \]

### 5.7 Analysis of service process

The service time follows PH distribution with representation \((\alpha, S)\) where

\[ \alpha = (1, 0, \ldots, 0)_{1 \times (m+n+2)a} \] and \( S = \begin{bmatrix} C_0' & C_1 & C_2 & 0 \\ C_3 & C_4' & C_5 & 0 \\ C_6 & 0 & C_7' & C_8 \\ 0 & 0 & 0 & C_9' \end{bmatrix}. \]

\[ C_0' = C_0 + \lambda I, \ C_4' = C_4 + \lambda I, \ C_7' = C_7 + \lambda I \] and \( C_9' = C_9 + \lambda I \). The absorbing state is represented by \( S^0 = B_2 \) which is a column matrix.

- The response time of the service process, \( E(S) = -\alpha S^{-1}e \).

- Hence the expected service rate \( \mu_s = \frac{1}{E(S)} \).

- **Theorem:** The queueing system is stable when \( \lambda < \mu_s \).
5.7. Analysis of service process

5.7.1 Stationary distribution

The stationary distribution, under the condition of stability, \( \lambda < \mu_s \) of the model, has Matrix Geometric solution. Let \( \chi = (x_0, x_1, x_2, \ldots) \) be the steady state probability vector of the Markov chain \( \{Z(t), t \geq 0\} \). Each \( x_i, i > 0 \) are vectors with \((2 + m + n)a\) elements. We assume that \( x_2 = x_1R \), and \( x_i = x_1R^{i-1}, i \geq 2 \), where \( R \) is the minimal non-negative solution to the matrix quadratic equation
\[
R^2A_2 + RA_1 + A_0 = 0.
\]

From \( \chi Q = 0 \) we get
\[
x_0B_0 + x_1B_2 = 0.
\]
\[
x_0B_1 + x_1(A_1 + RA_2) = 0.
\]

Solving the above two equations we get \( x_0 \) and \( x_1 \) subject to the normalizing condition \( x_0e + x_1(I - R)^{-1}e = 1 \).

Expected number of interruptions during the service of any customer

Let \( N'(t) \) be the number of interruptions due to first \( m \) environmental factors during the service of a particular customer at time \( t \). \( S(t) \) be the status of the server at time \( t \).

\[
S(t) = \begin{cases} 
0, & \text{when service is going on;} \\
1, & \text{if interrupted service going on;} \\
2, & \text{if server under repair}
\end{cases}
\]

\( I_1(t) \) is the environmental factor causing interruption and \( I_2(t) \) is the phase of service. Then \( \{(N'(t), S(t), I_1(t), I_2(t)), t \geq 0\} \) is a Markov chain with state space \( \{(r, 0, i) \cup (r, 1, j, i) \cup (r, 2, j, i); r = 1, 2, \ldots; i = 1, \ldots, a; j = 1, \ldots, n\} \).
Chapter 5. Queue with ignored interruption in random environment and self correction.

\[ \mathcal{Q} = \begin{bmatrix} 0 & 0 & 0 \\ U_2' & U_1' & U_0' \\ U_2 & 0 & U_1 & U_0 \\ U_2 & 0 & 0 & U_1 & U_0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \]

where \( U_2' \) is a column matrix of order \( a \times 1 \). \( U_2' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mu \end{bmatrix} \)

\[ U_1' = \begin{cases} -\beta \sum_{r=1}^{m} p_r - \mu, & \text{for } i = j; i, j = 1, \ldots, a \\ \mu, & \text{for } j = i + 1; i, j = 1, \ldots, a - 1 \\ 0, & \text{otherwise.} \end{cases} \]

\[ U_0' = (\beta p' \otimes I_a)_{a \times am}. \]

\( U_2 \) is a column matrix of order \((1 + 2m)a \times 1\) and

\[ U_2(i, 1) = \begin{cases} \mu_j, & \text{for } i = ar; r = 1, 2, \ldots, m; \\ \mu, & \text{for } i = a(2m + 1); \\ 0, & \text{otherwise.} \end{cases} \]

\[ U_1 = \begin{bmatrix} D_0 & D_1 & D_2 \\ 0 & D_3 & D_4 \\ 0 & 0 & D_5 \end{bmatrix}_{(1+2m)a \times (1+2m)a} \]

\[ D_0(i, j) = \begin{cases} \vartheta_r, & \text{for } i = j; i = ra + l; r = 0, \ldots, m - 1; l = 1, \ldots, a; \\ \mu_r, & \text{for } i + 1 = j; i = ra + l; r = 0, \ldots, m - 1; l = 1, \ldots, a - 1; \\ 0, & \text{otherwise.} \end{cases} \]

where \( \vartheta_r = -\mu_r - \gamma_r - \delta_r \).
5.8. Performance measures

Let $Z_k$ be the probability that there are exactly $k$ interruptions during the service of a customer due to first $m$ environmental factors. Then $Z_k = \begin{cases} \alpha(-U'_1)^{-1}U'_2, & \text{for } k = 0 \\ \alpha((-U'_1)^{-1}U'_0)\cdots(-U_1)^{-1}U_0)^{k-1}(-U_1)^{-1}U_2, & \text{for } k = 1, 2, 3, \ldots \end{cases}$

So the expected number of interruptions due to first $m$ environmental factors during single service $E(I) = \sum_{k=0}^{\infty} kZ_k$.

5.8 Performance measures

After finding the steady state probability vector we find the important performance measures of the system. The important measures are as follows.
Chapter 5. Queue with ignored interruption in random environment and self correction.

5.8.1 Expected waiting time

We consider the customer who joined as the $m^{th}$ in the queue. During the time of arrival of $m^{th}$ customer one customer in the system may be in service or the server may be in repair and other customers are waiting in the queue. So the waiting time of the tagged customer is the time until absorption of the Markov chain $W = \{(M(t), S(t), I_1(t), I_2(t)), t \geq 0\}$ where $M(t)$ is the rank of the tagged customer, $S(t), I_1(t)$ and $I_2(t)$ are as defined in earlier sections. The waiting time of the tagged customer follows phase type distribution with representation $(\omega, T)$ where

$$T = \begin{bmatrix}
S & S^0\alpha \\
S & S^0\alpha \\
S & S^0\alpha \\
... & ... & ... \\
... & ... & ... \\
... & ... & ...
\end{bmatrix}$$

$\omega$ is the initial probability vector.

Depending on the state of the server at the time of joining, the expected waiting time of the tagged customer, $E_W^m = -S^{-1}(I - (S^0\alpha S^{-1})^{(r-1)})(I - S^0\alpha S^{-1})^{-1}e$.

The expected waiting time of any customer who waits in the queue,

$$E(W) = \sum_{m=1}^{\infty} x_m E_W^m$$

5.8.2 Other important performance measures

- Probability that the system is idle, $P(I) = x_0$. 
5.9. Numerical Illustrations

- Probability that the system is working without interruption,
  \[ P(WI) = \sum_{i=1}^{\infty} (x_{i0}e + x_{i3}e) \]

- Probability that the system is under repair \( P(R) = \sum_{i=1}^{\infty} x_{i2}e \).

- Probability that the system is under protection \( P(p) = \sum_{i=1}^{\infty} x_{i3}e \).

- Expected number of customers in the system, \( E(C) = \sum_{i=1}^{\infty} ix_i e \).

- Effective interruption rate, \( E_{int} = \beta \sum_{i=1}^{\infty} x_{i1}e \).

- Effective rate of self correction, \( E_{selfcorr} = \sum_{i=1}^{\infty} \sum_{j=1}^{m} \delta_j x_{ij}e \).

- Effective rate of protection, \( E_{protection} = \sum_{i=1}^{\infty} \sum_{j=m+1}^{n} \eta_j x_{ij}e \).

5.9 Numerical Illustrations

In this section assuming arbitrary values for the parameters, subject to stability, we obtained the numerical values for important performance measures. Let \( n = 4, m = 2, \mu = 7, \mu_1 = 5, \mu_2 = 4; \eta_1 = 4, \eta_2 = 3, \eta_3 = 2, \eta_4 = 1, \beta = .5; \gamma_1 = 1, \gamma_2 = 2; \delta_1 = 1, \delta_2 = .5; p_1 = p_2 = p_3 = p_4 = 0.25 \).

The conclusion drawn are purely based on the values of input parameters.
Effect of $\lambda$ on various performance measures

Table 5.5: Effect of $\lambda$ on various performance measures

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$E(C)$</th>
<th>$P(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9230</td>
<td>0.5012</td>
</tr>
<tr>
<td>1.5</td>
<td>1.9157</td>
<td>0.3134</td>
</tr>
<tr>
<td>2</td>
<td>3.5376</td>
<td>0.1910</td>
</tr>
<tr>
<td>2.5</td>
<td>6.1867</td>
<td>0.1166</td>
</tr>
<tr>
<td>3</td>
<td>10.5083</td>
<td>0.0716</td>
</tr>
<tr>
<td>3.5</td>
<td>17.1259</td>
<td>0.0443</td>
</tr>
<tr>
<td>4</td>
<td>24.7322</td>
<td>0.0276</td>
</tr>
</tbody>
</table>

As the arrival rate $\lambda$ increases the expected number of customers in the system $E(C)$ increase, but probability for idleness of the server $P(I)$ decrease which are on expected lines (refer Table 5.5).

Effect of $\mu$ on various performance measures

Assuming $\lambda = 2$ and varying $\mu$ we get the following values for different performance measures.

As the initial service rate $\mu$ increases the expected service time $E(s)$, the expected number of customers in the system $E(C)$, probability for repair $P(R)$, expected rate of interruption $E_{int}$ and expected rate of self correction $E_{selfcorr}$ decrease but probability for idleness of the server $P(I)$ increase which are on expected lines. $\mu$ increases means number of service completion in unit time increases. So rate of self correction, rate of interruption and probability for repair in unit time reduces (see Table 5.6).
5.9. Numerical Illustrations

Table 5.6: Effect of $\mu$ on various performance measures

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$E(S)$</th>
<th>$E(C)$</th>
<th>$P(I)$</th>
<th>$P(R)$</th>
<th>$E_{int}$</th>
<th>$E_{selfcorr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.2897</td>
<td>29.2040</td>
<td>0.0175</td>
<td>0.1422</td>
<td>0.3322</td>
<td>0.0983</td>
</tr>
<tr>
<td>4</td>
<td>1.7508</td>
<td>15.5352</td>
<td>0.0471</td>
<td>0.1437</td>
<td>0.3364</td>
<td>0.0978</td>
</tr>
<tr>
<td>5</td>
<td>1.4180</td>
<td>8.0176</td>
<td>0.0891</td>
<td>0.1398</td>
<td>0.3281</td>
<td>0.0938</td>
</tr>
<tr>
<td>6</td>
<td>1.1917</td>
<td>5.0105</td>
<td>0.1384</td>
<td>0.1338</td>
<td>0.3147</td>
<td>0.0885</td>
</tr>
<tr>
<td>7</td>
<td>1.0279</td>
<td>3.5376</td>
<td>0.1910</td>
<td>0.1267</td>
<td>0.2986</td>
<td>0.0827</td>
</tr>
<tr>
<td>8</td>
<td>0.9037</td>
<td>2.6963</td>
<td>0.2437</td>
<td>0.1192</td>
<td>0.2814</td>
<td>0.0769</td>
</tr>
<tr>
<td>9</td>
<td>0.8063</td>
<td>2.1620</td>
<td>0.2946</td>
<td>0.1117</td>
<td>0.2641</td>
<td>0.0713</td>
</tr>
</tbody>
</table>

Effect of $\beta$ on various performance measures

Table 5.7: Effect of $\beta$ on various performance measures

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$E(S)$</th>
<th>$E(C)$</th>
<th>$P(I)$</th>
<th>$P(R)$</th>
<th>$E_{int}$</th>
<th>$E_{selfcorr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>1.4180</td>
<td>8.0176</td>
<td>0.0891</td>
<td>0.1398</td>
<td>0.3281</td>
<td>0.0938</td>
</tr>
<tr>
<td>1</td>
<td>1.5863</td>
<td>11.0407</td>
<td>0.0688</td>
<td>0.2230</td>
<td>0.5230</td>
<td>0.1502</td>
</tr>
<tr>
<td>2</td>
<td>1.8247</td>
<td>16.4010</td>
<td>0.0476</td>
<td>0.3157</td>
<td>0.7394</td>
<td>0.2140</td>
</tr>
<tr>
<td>3</td>
<td>1.9818</td>
<td>20.3997</td>
<td>0.0373</td>
<td>0.3648</td>
<td>0.8539</td>
<td>0.2484</td>
</tr>
<tr>
<td>4</td>
<td>2.0906</td>
<td>23.1333</td>
<td>0.0314</td>
<td>0.3946</td>
<td>0.9231</td>
<td>0.2695</td>
</tr>
<tr>
<td>5</td>
<td>2.1691</td>
<td>24.9636</td>
<td>0.0277</td>
<td>0.4141</td>
<td>0.9685</td>
<td>0.2834</td>
</tr>
</tbody>
</table>

From Table 5.7 we note that as the interruption rate $\beta$ increases effective service time $E(S)$, the expected number of customers in the system $E(S)$, probability for repair $P(R)$, expected rate of interruption $E_{int}$ and expected rate of self correction $E_{selfcorr}$ increases but probability for idleness of the server $P(I)$ decrease which are on expected lines.
Effect of $\gamma$ on various performance measures

Assuming $\gamma_1 = \gamma_2 = \gamma$ and varying over its value we get the following table for different performance measures. As the interruption clock realization rate $\gamma$ increases effective service time $E(S)$, the expected number of customers in the system $E(C)$, probability for repair $P(R)$, expected rate of interruption $E_{int}$ increase but probability for idleness of the server $P(I)$ and expected rate of self correction $E_{selfcorr}$ decrease which are on expected lines. Rate of protection decrease with increase in $\gamma$. As the realization rate of interruption clock increase the server immediately goes for repair reducing the chance for self correction (see Table 5.8).

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$E(S)$</th>
<th>$E(C)$</th>
<th>$P(I)$</th>
<th>$P(R)$</th>
<th>$E_{int}$</th>
<th>$E_{selfcorr}$</th>
<th>$E_{protection}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.4069</td>
<td>7.7491</td>
<td>0.0915</td>
<td>0.1303</td>
<td>0.3254</td>
<td>0.1184</td>
<td>0.1627</td>
</tr>
<tr>
<td>1</td>
<td>1.4136</td>
<td>7.9371</td>
<td>0.0898</td>
<td>0.1367</td>
<td>0.3283</td>
<td>0.1030</td>
<td>0.1642</td>
</tr>
<tr>
<td>1.5</td>
<td>1.4192</td>
<td>8.0705</td>
<td>0.0887</td>
<td>0.1413</td>
<td>0.3302</td>
<td>0.0919</td>
<td>0.165</td>
</tr>
<tr>
<td>2</td>
<td>1.4240</td>
<td>8.1718</td>
<td>0.0878</td>
<td>0.1448</td>
<td>0.3315</td>
<td>0.0832</td>
<td>0.1657</td>
</tr>
<tr>
<td>2.5</td>
<td>1.4282</td>
<td>8.2523</td>
<td>0.0871</td>
<td>0.1475</td>
<td>0.3324</td>
<td>0.0763</td>
<td>0.1662</td>
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<tr>
<td>3</td>
<td>1.4318</td>
<td>8.3184</td>
<td>0.0865</td>
<td>0.1498</td>
<td>0.3331</td>
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<td>3.5</td>
<td>1.4349</td>
<td>8.3739</td>
<td>0.0860</td>
<td>0.1517</td>
<td>0.33367</td>
<td>0.0656</td>
<td>0.1668</td>
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</tbody>
</table>