Chapter 7

Stochastic decomposition of the M/M/1 queue with environment dependent working vacation

Introduction

If a queue is empty the server remains idle. Policy we may think of a vacation with server working during that time in slow mode, if customers are available. The idle time of the server can be utilized for some other work. Instead of a complete vacation if the customers in the queue is less, the functioning of the server in a slow rate will reduce the operating cost, energy consumption and the startup cost. These advantages are pointing
towards working vacation. Working vacation is an extension of regular vacation. In working vacation, instead of completely stopping the service, the server provides service at a slow rate. Working vacation reduces the chance of reneging of the customers compared to normal vacation. In this era of high demand for commodities and services which are available in a short spell, the concept of working vacation is very useful. This may be the main reason of the extensive research work going on in working vacation queueing models.

In this chapter we consider a single server queueing system with working vacation. On completion of a service if the server finds the system empty, he goes for a working vacation. There are \( n \) types of working vacations. Depending on the environment, after a busy period, the server goes for \( i^{th} \) type of vacation with probability \( p_i, 1 \leq i \leq n \). During vacation if customers arrive, the server provides service at a lower rate. On completion of service during vacation, if there is no customer in the system the server continues to be on vacation. Otherwise the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service in the normal rate. On completion of vacation if the server finds the system empty, he remains in the corresponding vacation.

We demonstrate stochastic decomposition of the queue length and waiting time processes using method of induction and Little’s formula.

### 7.1 Model description

Consider a single server queueing system with working vacation in which arrival occurs according to a Poisson process with parameter \( \lambda \). The service time is exponentially distributed with parameter \( \mu \). On completion
of a service if the server finds the system empty he goes for a working vacation. There are $n$ types of working vacations. Depending on the environment, after a busy period, the server goes for $i^{th}$ type of vacation with probability $p_i, 1 \leq i \leq n$. The duration of $i^{th}$ type of vacation is exponentially distributed with parameter $\gamma_i, 1 \leq i \leq n$. During vacation if customers arrive, the server provides service at a lower rate $\mu_i$, while in $i^{th}$ type of vacation, $1 \leq i \leq n$. On completion of service during vacation, if there is no customer in the system the server continues to stay on vacation. Otherwise the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service in the normal rate $\mu$. On completion of vacation if the server finds the system empty, he remains in the corresponding vacations. Figure 7.1 is a diagrammatic representation of the model.
7.2 Mathematical description

We establish the stochastic decomposition of the state space by induction on the number of environmental factors.

**Case 1** First we consider the case of $n = 2$.

Let $N(t)$ be the number of customers in the system and $S(t)$ be the status of the server at time $t$:

$$S(t) = \begin{cases} 
0, & \text{if the server is serving in normal mode;} \\
1, & \text{if server is in the type I working vacation;} \\
2, & \text{if server is in the type II working vacation;}
\end{cases}$$

Then $X = \{X(t), t \geq 0\}$ where $X(t) = (N(t), S(t))$ is a continuous time Markov chain with state space $\{0, 1\} \cup \{0, 2\} \cup \{(j, k), j = 1, 2, \ldots; k = 0, 1, 2\}$. The infinitesimal generator associated with the Markov chain is

$$Q_1 = \begin{bmatrix} B_0 & B_1 \\
B_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots \end{bmatrix}$$

where $-B_0 = A_0 = \lambda I$.

$$B_1 = \begin{bmatrix} 0 & \lambda & 0 \\
0 & 0 & \lambda \end{bmatrix}, \quad B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 \\
\mu_1 & 0 \\
0 & \mu_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mu & 0 & 0 \\
\mu_1 & 0 & 0 \\
\mu_2 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda - \mu & 0 & 0 \\
\theta_1 & -\lambda - \mu_1 - \theta_1 & 0 \\
\theta_2 & 0 & -\lambda - \mu_2 - \theta_2 \end{bmatrix}$$
7.2. Mathematical description

Stability analysis

We have
\[ A = A_0 + A_1 + A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \theta_1 + \mu_1 & -\mu_1 - \theta_1 & 0 \\ \theta_2 + \mu_2 & 0 & -\mu_2 - \theta_2 \end{bmatrix} \]

Then \( A \) is the infinitesimal generator of a Markov chain with state space \( \{0, 1, 2\} \) which represents the status of the server. Let \( y = (y_0, y_1, y_2) \) be the invariant probability vector of \( A \). Then \( yA = 0 \) and \( ye = 1 \). The left drift rate of the original Markov chain is \( yA_2e \) and that for right drift is \( yA_0e \). Left drift indicates a service completion and right drift represents arrival of customer. Thus the system is stable if and only if \( yA_0e < yA_2e \).

Here \( yA_0e = \lambda \) and \( yA_2e = \mu \).

Hence we have

Theorem: The system is stable if and only if \( \lambda < \mu \).

7.2.1 Steady State Analysis

For the analysis of the model it is necessary to solve for the minimal non-negative solution \( R_1 \) of the matrix quadratic equation

\[ R_1^2 A_2 + R_1 A_1 + A_0 = 0. \]  \hspace{1cm} (7.1)

Since the Matrices \( A_2, A_1, A_0 \) are lower triangular \( R_1 \) is also lower triangular. Solving (7.1) we obtain \( R_1 \) as

\[ R_1 = \begin{bmatrix} r_0 & 0 & 0 \\ r_1 & r_1 & 0 \\ r_2 & 0 & r_2 \end{bmatrix} \]

where \( r_0 = \rho \),

\[ r_1 = \frac{\rho(\lambda + \theta_1)}{(\lambda + \mu_1 + \theta_1)}, \quad r_1 = \frac{\lambda}{(\lambda + \mu_1 + \theta_1)}, \quad r_2 = \frac{\rho(\lambda + \theta_2)}{(\lambda + \mu_2 + \theta_2)}, \quad r_2 = \frac{\lambda}{(\lambda + \mu_2 + \theta_2)} \]

Let \( x = (x_0, x_1, x_2, \ldots) \) be the steady state probability vector associated
with the Markov process $X$. Here $x_0 = (x_{01}, x_{02})$ and $x_i = (x_{i0}, x_{i1}, x_{i2}), i = 1, 2, \ldots, \infty$. Assume that $x_i = x_1 R_1^{i-1}, i = 2, 3, \ldots$, then $x$ can be obtained by solving $xQ = 0$ using the boundary condition

$$x_0 e + x_1 (I - R_1)^{-1} e = 1. \quad (7.2)$$

From $xQ = 0$ we get

$$x_0 B_0 + x_1 B_2 = 0. \quad (7.3)$$

$$x_0 B_1 + x_1 (A_1 + R_1 A_2) = 0. \quad (7.4)$$

From (7.3) and (7.4) we will get

$$\mu p_1 x_{10} + \mu_1 x_{11} = (\lambda)x_{01}. \quad (7.5)$$

$$\mu p_2 x_{10} + \mu_2 x_{12} = (\lambda)x_{02}. \quad (7.6)$$

$$\mu x_{10} = (\lambda + \theta_1)x_{11} + (\lambda + \theta_2)x_{12}. \quad (7.7)$$

$$\lambda x_{01} = (\lambda + \mu_1 + \theta_1)x_{11}. \quad (7.8)$$

$$\lambda x_{02} = (\lambda + \mu_2 + \theta_2)x_{12}. \quad (7.9)$$

Assume $x_{01} = k_1$ and $x_{02} = k_2$, then from (7.8) and (7.9), $x_{11} = \tau_1 k_1$, $x_{12} = \tau_2 k_2$. Substituting the values of $x_{11}$ and $x_{01}$ in (7.5) we will get $x_{10} = \frac{k_1 \tau_1}{\mu_1}$. Also

$$k_2 = \frac{\mu_2 \tau_1}{p_1 (\lambda - \mu_2 \tau_2)} k_1$$

To find the value of $k_1$ we use the normalizing condition $x_0 e + x_1 (I - R_1)^{-1} e = 1$. 

Let \( r'_0 = 1-r_0, r'_1 = 1-r_1, r'_2 = 1-r_2; \) then \((I-R_1)^{-1} = \)

\[
\begin{bmatrix}
\frac{1}{r'_0} & 0 & 0 \\
-r_1/r'_0' & \frac{1}{r'_1} & 0 \\
-r_2/r'_0' & 0 & \frac{1}{r'_2}
\end{bmatrix}
\]

Using (7.2)

\[
k_1 \left[ 1 + \frac{r_1}{p_1 r'_0} + \frac{r_1}{r'_1} - \frac{r_1 r_1}{r'_0 r'_1} \right] + k_2 \left[ 1 + \frac{r_2}{r'_2} - \frac{r_2 r_1}{r'_0 r'_2} \right] = 1.
\]

(7.10)

Substituting \( k_2 \) in (7.10)

\[
k_1 \left[ 1 + \frac{r_1}{p_1 r'_0} + \frac{r_1}{r'_1} - \frac{r_1 r_1}{r'_0 r'_1} \right] + \frac{1}{p_1 (\lambda - \mu r_1)} \left[ 1 + \frac{r_2}{r'_2} - \frac{r_2 r_1}{r'_0 r'_2} \right] = 1.
\]

(7.11)

From (7.11) \( k_1 = \)

\[
\left[ \begin{array}{ccc}
\frac{1}{r_0} & 0 & 0 \\
\frac{r_1}{p_1 r'_0} & \frac{1}{r'_1} & 0 \\
\frac{r_2}{r'_0 r'_2} & 0 & \frac{1}{r'_2}
\end{array} \right]
\]

Now \( R_1^{-1} = \)

\[
\left[ \begin{array}{ccc}
\frac{1}{r_0} & 0 & 0 \\
\frac{r_1}{p_1 r'_0} & \frac{1}{r'_1} & 0 \\
\frac{r_2}{r'_0 r'_2} & 0 & \frac{1}{r'_2}
\end{array} \right]
\]

and

\[
x_{k} = x_{10} k_1 + x_{11} \left[ \frac{1}{r'_1} + r_1 \frac{r_1}{p_1 (\lambda - \mu r_1)} \right] + x_{12} \left[ \frac{1}{r'_2} + r_2 \frac{r_1}{p_1 (\lambda - \mu r_1)} \right]
\]

for \( k > 1. \)

Let \( Q_v(z) \) be the PGF associated with the number of customers in the system. Then \( Q_v(z) = \sum_{n=0}^{\infty} x_{n} e_{n} z^{n} \)

\[
= x_{01} + x_{02} + x_{10} \frac{z}{1-z} + x_{11} \frac{z}{1-z} + x_{12} \frac{z}{r_0 - r_1} \left[ \frac{1}{1-r_0} - \frac{1}{1-r_1} \right] + x_{12} \frac{z}{r_0 - r_2} \left[ \frac{1}{1-r_0} - \frac{1}{1-r_2} \right]
\]

\[
= 1 - \frac{r_0}{1-r_0} \left[ x_{01} \frac{(1-r_0)}{1-r_0} + x_{02} \frac{(1-r_0)}{1-r_0} + x_{10} \frac{z}{r_0 - r_1} + x_{11} \frac{z}{r_0 - r_1} + x_{12} \frac{z}{r_0 - r_2} + x_{12} \frac{z}{r_0 - r_2} \right]
\]

\[
Q_v(z) = \frac{r_0}{1-r_0} \left[ x_{01} \frac{(1-r_0)}{1-r_0} + x_{02} \frac{(1-r_0)}{1-r_0} + x_{10} \frac{z}{r_0 - r_1} + x_{11} \frac{z}{r_0 - r_1} + x_{12} \frac{z}{r_0 - r_2} + x_{12} \frac{z}{r_0 - r_2} \right]
\]

\[
\left[ x_{10} \frac{x_{1}}{r_0 - r_1} + x_{11} \frac{z}{r_0 - r_1} + x_{12} \frac{z}{r_0 - r_2} + x_{12} \frac{z}{r_0 - r_2} \right]
\]
The state space of $X$ is $\{(0,k) | k = 1, 2, 3\} \cup \{(j,k) | j = 1, 2, \ldots; k = 0,1,2,3\}$. The infinitesimal generator associated with the Markov chain is $Q_2$.

\[
x_{12} \left( \frac{r_0 - r_2 - r_3}{r_0 - r_2} \right) \left( \frac{1 - 2r_0 + r_0 p_2}{1 - r_2} \right)^2 \]

Expected queue length $E(L) = Q'_2(1) = \frac{r_0}{1-r_0} + \left( \frac{1}{1-r_0} \right) \left( \frac{r_0}{1-r_0} \right) \left( \frac{1 - 2r_0 + r_0 p_2}{1 - r_2} \right)^2 + \left( \frac{r_0}{1-r_0} \right) \left( \frac{r_0}{1-r_0} \right) \left( \frac{1 - 2r_0 + r_0 p_2}{1 - r_2} \right)^2 + \left( \frac{r_0}{1-r_0} \right) \left( \frac{r_0}{1-r_0} \right) \left( \frac{1 - 2r_0 + r_0 p_2}{1 - r_2} \right)^2 + \left( \frac{r_0}{1-r_0} \right) \left( \frac{r_0}{1-r_0} \right) \left( \frac{1 - 2r_0 + r_0 p_2}{1 - r_2} \right)^2.
\]

Case 2

Now consider the case of $n = 3$. Then $S(t)$ has four states:

\[
S(t) = \begin{cases} 
0, & \text{if the server is serving in normal mode;} \\
1, & \text{if server is in the type I working vacation;} \\
2, & \text{if server is in the type II working vacation;} \\
3, & \text{if server is in the type III working vacation.}
\end{cases}
\]

The infinitesimal generator associated with the Markov chain is:

\[
Q_2 = \begin{bmatrix} 
B_0 & B_1 \\
B_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots 
\end{bmatrix}
\]
where \(-B_0 = A_0 = \lambda I_3\), \(B_1 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & \lambda \end{bmatrix}\), \(B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 & \mu p_3 \\ \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \end{bmatrix}\),

\[
A_2 = \begin{bmatrix} \mu & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_2 & 0 & 0 \\ \mu_3 & 0 & 0 \end{bmatrix},
\]

\[
A_1 = \begin{bmatrix} -\lambda - \mu & 0 & 0 & 0 \\ \theta_1 & -\lambda - \mu_1 - \theta_1 & 0 & 0 \\ \theta_2 & 0 & -\lambda - \mu_2 - \theta_2 & 0 \\ \theta_3 & 0 & 0 & -\lambda - \mu_3 - \theta_3 \end{bmatrix}.
\]

\[
A = A_0 + A_1 + A_2 = \begin{bmatrix} r_0 & 0 & 0 & 0 \\ r_1 & r_1' & 0 & 0 \\ r_2 & 0 & r_2' & 0 \\ r_3 & 0 & 0 & r_3' \end{bmatrix}
\]

where \(r_0 = \rho, r_1 = \frac{\rho(\lambda+\theta_1)}{(\lambda+\mu_1+\theta_1)}, r_1' = \frac{\lambda}{(\lambda+\mu_1+\theta_1)}\),

\[
r_2 = \frac{\rho(\lambda+\theta_2)}{(\lambda+\mu_2+\theta_2)}, r_2' = \frac{\lambda}{(\lambda+\mu_2+\theta_2)}, r_3 = \frac{\rho(\lambda+\theta_3)}{(\lambda+\mu_3+\theta_3)} \text{ and } r_3' = \frac{\lambda}{(\lambda+\mu_3+\theta_3)}.
\]

Let \(x = (x_0, x_1, x_2, \ldots)\) be the steady state probability vector associated with the Markov process \(X\). Here \(x_0 = (x_{01}, x_{02}, x_{03})\) and \(x_i = (x_{0i}, x_{1i}, x_{2i}, x_{3i}), i = 1, 2, \ldots\). Then assuming \(x_{01} = k_1, x_{02} = k_2\) and \(x_{03} = k_3\), we get \(x_{11} = r_1 k_1\), \(x_{12} = r_2 k_2\), \(x_{13} = r_3 k_3\), \(x_{10} = \frac{k_1 r_1}{p_1}\). Also \(k_2 = \frac{\mu_2 r_2}{p_1(\lambda-\mu_2 r_2)} k_1\), \(k_3 = \frac{\mu_3 r_3}{p_1(\lambda-\mu_3 r_3)} k_1\).

Let \(r_0' = 1 - r_0, r_1' = 1 - r_1, r_2' = 1 - r_2, r_3' = 1 - r_3\) then
Now we consider the case where there are $n \geq 4$ distinct type of vacations.

Using the normalizing condition $x_0 e + x_1 (I - R_2)^{-1} e = 1$, we get

$$k_1 = \left[ \begin{array}{cccc} 1 + \frac{1}{r_1} - \frac{1}{r_1} + \frac{1}{r_1} & \sum_{j=2}^{3} \frac{\mu p_j r_1}{p_1 (\lambda - \mu_j r_j)} \left( 1 + \frac{r_j}{r_j} - \frac{r_j r_j}{r_0 r_0} \right) \\ 1 \end{array} \right]$$

and

$$R_2^{k-1} = \left[ \begin{array}{cc} 1 \end{array} \right]$$

Case. 3

Now we consider the case where there are $n \geq 4$ distinct type of vacations.
7.2. Mathematical description

Then $S(t)$ has $n + 1$ distinct values.

$$S(t) = \begin{cases} 0, & \text{if the server is serving in normal mode;} \\ i, & \text{if server is in the } i\text{th type working vacation, } 1 \leq i \leq n; \end{cases}$$

The state space of $X$ is $\{(0, k)/k = 1, 2, \ldots, n\} \cup \{(j, k)/j = 0, 1, 2, \ldots; k = 1, 2, \ldots, n\}$ The infinitesimal generator associated with the Markov chain is

$$Q_n = \begin{bmatrix} B_0 & B_1 & \cdots & \cdots & \cdots & \cdots \\ B_2 & A_1 & A_0 & \cdots & \cdots & \cdots \\ \vdots & A_2 & A_1 & A_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad \text{where } B_1 = \begin{bmatrix} 0 & \lambda & & & & \\ & & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & \\ & & & & & \lambda \end{bmatrix}_{n \times (n+1)},$$

$$B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 & \cdots & \mu p_n \\ \mu_1 & & & \cdots & \cdots \\ & \mu_2 & & \cdots & \cdots \\ & & \mu_3 & \cdots & \cdots \\ & & & \ddots & \cdots \end{bmatrix}_{(n+1) \times n},$$

$$A_2 = \begin{bmatrix} \mu & & & & \\ \mu_1 & \cdots & & & \\ & \cdots & \cdots & \cdots & \cdots \\ & & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots \\ & & & & \mu_n \end{bmatrix}_{(n+1) \times (n+1)},$$

and $-B_0 = A_0 = \lambda I_n$

$$A_1 = \begin{bmatrix} -\lambda - \mu & \theta_1 & -\lambda - \mu - \theta_1 \\ \theta_2 & -\lambda - \mu_1 - \theta_1 & -\lambda - \mu_2 - \theta_2 \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots \\ \theta_n & \vdots & \cdots \end{bmatrix}_{\cdots}$$

As in the earlier sections
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environment dependent working vacation

Let \( y = (y_0, y_1, y_2, \ldots, y_n) \) be the invariant probability vector of \( A \) satisfying \( yA = 0 \) and \( ye = 1 \). The system is stable if and only if \( yA_0e < yA_2e \).

Theorem: The system is stable if and only if \( \lambda < \mu \)

\[
R_n = \begin{bmatrix}
  r_0 & r_1 & r_2 & \cdots & r_n \\
  r_1 & \tau_1 & 0 & \cdots & 0 \\
  r_2 & 0 & \tau_2 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  r_n & 0 & \cdots & \cdots & \tau_n \\
\end{bmatrix}
\]

where \( r_0 = \rho, r_i = \frac{\rho(\lambda+\theta)}{\lambda+\mu_i+\theta}, \) \( \tau_i = \frac{\lambda}{(\lambda+\mu_i+\theta)}, \) \( 1 \leq i \leq n, \) \( x_0 = x_01, x_02, \ldots, x_0n \) and \( x_i = (x_{i0}, x_{i1}, x_{i2}, \ldots, x_{in}), i = 1, 2, \ldots \). Then assuming \( x_{0j} = k_j, 1 \leq i \leq n, \) we get \( x_{1j} = \tau_j k_j, x_{10} = k_1 \tau_1 k_1 \).

Also \( k_j = \frac{pp_j \tau_1}{\mu_j (\lambda-\mu_j \tau_j)} k_1 \). Let \( \tau'_i = 1 - \tau_i, 1 \leq i \leq n, r'_0 = 1 - r_0, \)
\( \ell_i = 1/\tau'_i, 0 \leq i \leq n, \chi_i = -r_i/(\tau'_i \tau'_0), 1 \leq i \leq n \) then,
\[(I - R_n)^{-1} = \begin{bmatrix}
\ell_0 \\
\chi_1 & \ell_2 \\
\chi_2 & \ell_2 \\
\vdots & \vdots \\
\chi_n & \ell_n
\end{bmatrix}\]

\[k_1 = \frac{r_1}{\lambda - \mu} + \frac{r_0}{\mu} + \sum_{j=2}^{n} \frac{\mu p_j r_1}{p_1 \lambda - \mu_j r_j} \left(1 + \frac{r_j}{\tau_j} - \frac{r_j \tau_j}{r_j \tau_j}ight)\]

Now \[R_n^{k-1} = \begin{bmatrix}
p_0^{(k-1)} \\
p_1^{(k-1)} & r_1 \left(\frac{r_0^{k-1} - r_1^{k-1}}{r_0 - r_1}\right) \quad \tau_1^{(k-1)} \\
p_2^{(k-1)} & r_2 \left(\frac{r_0^{k-1} - r_2^{k-1}}{r_0 - r_2}\right) \quad 0 \quad \tau_2^{(k-1)} \\
\vdots & \vdots & \ddots & \ddots \\
r_n^{(k-1)} & \left(\frac{r_0^{k-1} - r_n^{k-1}}{r_0 - r_n}\right) & \cdots & \cdots & \tau_n^{(k-1)}
\end{bmatrix}\]

\[x_k e = x_10 r_0^{k-1} + \sum_{i=1}^{n} x_{1i} \left[\tau_i^{(k-1)} + r_i \left(\frac{r_0^{k-1} - \tau_i^{k-1}}{r_0 - \tau_i}\right)\right]\text{ for } k > 1.\]

Then \[Q_v(z) = \sum_{n=0}^{\infty} x_n z^n\]

\[= \sum_{j=1}^{n} x_{0j} + \frac{x_10 z}{1 - r_0 z} + \sum_{j=1}^{n} \frac{x_{1j} z}{1 - \tau_j z} + \sum_{j=1}^{n} x_{1j} r_j z \left[1 - \frac{1}{1 - r_0 z} - \frac{1}{1 - \tau_j z}\right]\]

Expected queue length \[E(L) = Q_v'(1)\]

\[= \frac{r_0}{1 - r_0} + \sum_{j=1}^{n} \left(\frac{k_j}{1 - r_0}\right) \left[-r_0 + \frac{r_j \tau_j}{p_1} + \frac{r_j \tau_j}{(1 - \tau_j)(r_0 - \tau_j)} + \frac{r_j \tau_j}{r_0 - \tau_j} \left(\frac{1 - 2 \tau_j + \tau_j}{(1 - \tau_j)^2}\right)\right].\]

The above discussions lead to

**Theorem (Stochastic decomposition):** The expected queue length \[E(L)\] can be decomposed into the sum of the expectations of \(n + 1\) independent random variables as: \[E(L) = E(L) + \sum_{i=1}^{n} E(L_{V_i})\text{ where } E(L)\]
is the queue length of classical $M/M/1$ queue and $\sum_{i=1}^{n} E(L_{V_i})$ is the additional queue length due to $n$ types of vacations.

### 7.2.2 Stationary waiting time

Using Little’s formula the expected waiting time $E(W) = \frac{E(L)}{\lambda}$.

$$E(W) = \left( \frac{1}{\mu - \lambda} + \frac{1}{\lambda} \sum_{i=1}^{n} E(L_{V_i}) \right) \quad (7.12)$$

From (7.12) it is clear that the expected waiting time can be decomposed into the sum of $n + 1$ independent random variables: $E(W) = E(W) + \sum_{i=1}^{n} E(W_{V_i})$, where $E(W)$ is the expected waiting time of a customer in the $M/M/1$ queue and $\sum_{i=1}^{n} E(L_{W_i})$ is the additional waiting time due to $n$ types of vacations.