Chapter 2

Diameter variability of the product graphs

The diameter of a graph can be affected by the addition or the deletion of edges. In this chapter we examine the product graphs whose diameter increases (decreases) by the deletion (addition) of a single edge. The problems of minimality and maximality of the product graphs with respect to its diameter are also solved. These problems are motivated by the fact that most of the graph products are good interconnection networks and a good network must be hard to disrupt and the transmis-

Some results of this chapter are included in the following paper.  
sions must remain connected even if some vertices or edges fail.

2.1 Diameter variability of the Cartesian product of graphs

If both $H_1$ and $H_2$ are $K_2$'s, then $G$ is $C_4$ and the deletion of any edge increases the diam($G$).

**Theorem 2.1.1.** Let $G \cong H_1 \Box H_2$. Then $D^0(G) \geq 2$.

**Proof.** We shall prove the theorem by showing that there exist at least two edges in $G$ that can be deleted without an increase in the diam($G$) by considering the following three cases.

**Case 1:** $H_1$ and $H_2$ are complete graphs where $n_1$ or $n_2 > 2$.

Suppose that both $n_1, n_2 > 2$.

Let the two edges $u_iv_p - u_iv_q$ and $u_jv_r - u xv_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There are paths of length two between $u_iv_p, u_iv_q$ and $u_jv_r, u xv_r$ in $G$. Now, consider the vertices whose diametral path contain the deleted edges. The distance between these vertices remains the same, since $\delta(G) \geq 4$ there is an alternate path
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of length $\text{diam}(G)$ through the neighbours of the deleted edge. Also, the distance between any two other vertices is not affected by the removal of these two edges.

Suppose that $n_1 = 2$ and $n_2 > 2$. Let the two edges $u_1 v_p - u_1 v_q$ and $u_2 v_q - u_2 v_r$ where $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There are paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these two edges. Thus, the $\text{diam}(G)$ remains the same.

**Case 2:** $H_1$ and $H_2$ are not complete graphs.

Let the two edges $u_i v_p - u_i v_q$ and $u_j v_r - u_x v_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There is a path $u_i v_p - u_y v_p - u_y v_q - u_i v_q$ of length three between $u_i v_p$ and $u_i v_q$. Similarly, $d(u_j v_r, u_x v_r) \leq 3$. Now, consider the vertices whose diametral path contain the deleted edges. The distance between these vertices remains the same, since $\delta(G) \geq 2$ there is an alternate path of length $\text{diam}(G)$ through the neighbours of the deleted edge. Thus, the $\text{diam}(G)$ remains the same.
Case 3: \( H_1 \) is a complete graph and \( H_2 \) is a not complete graph.

Let the two edges \( u_i v_p - u_j v_p \) and \( u_i v_q - u_j v_q \) where \( i \neq j \in \{1, 2, \ldots, n_1\} \) and \( v_p \) is not adjacent to \( v_q \) in \( H_2 \), \( p, q \in \{1, 2, \ldots, n_2\} \), be deleted. There is a path of length at most three between these pairs of vertices. Therefore, \( d(u_i v_p, u_i v_q) \leq 3 \) and \( d(u_i v_q, u_j v_q) \leq 3 \). Also, the distance between any two other vertices is not affected by the removal of these two edges. Thus, the diam\( (G) \) remains the same.

Hence, there exist at least two edges in \( G \) that can be deleted without an increase in the diam\( (G) \). \( \square \)

**Theorem 2.1.2.** Let \( G \cong H_1 \square H_2 \). Then \( D^0(G) = 2 \) if and only if \( G \) is any one of the graphs shown in Fig 2.1.

![Fig 2.1: The graphs \( G : D^0(G) = 2 \).](image)

Proof. Suppose that \( G \) is any one of the graphs shown in Fig 2.1, then by deleting the bold edges, it is clear that \( D^0(G) = 2 \).
Conversely suppose that $D^0(G) = 2$. We shall show that $G$ is precisely any one of the graphs in Fig 2.1.

Let $u_x, u_y$ be a pair of diametral vertices in $H_1$, by a path $u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y$ and $v_w, v_z$ be a pair of diametral vertices in $H_2$, by a path $v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z$.

Let $G \cong K_{n_1} \square K_{n_2}$ where $n_1, n_2 > 2$.

Let the three edges $u_iv_p - u_iv_q, u_jv_q - u_jv_r$ and $u_xv_p - u_xv_r$ where $i \neq j \neq x \in \{1, 2, \ldots, n_1\}$ and $p \neq q \neq r \in \{1, 2, \ldots, n_2\}$, be deleted. There is a path $u_iv_p - u_iv_r - u_iv_q$ of length two between $u_iv_p$ and $u_iv_q$ in $G$ and so $d(u_iv_p, u_iv_q) = 2$. Similarly, $d(u_jv_q, u_jv_r) = d(u_xv_p, u_xv_r) = 2$. Also, the distance between any two other vertices is not affected by the removal of these three edges. Thus, the diam($G$) remains the same.

Let $G \cong H_1 \square H_2$, where $H_1$ and $H_2$ are not complete graphs.

Let the three edges $u_iv_p - u_jv_p, u_iv_q - u_jv_q$ and $u_av_p - u_av_r$ where $i \neq j \neq a \in \{1, 2, \ldots, n_1\}$ and $v_p$ is not adjacent to $v_q$ in $H_2$, $p, q \neq r \in \{1, 2, \ldots, n_2\}$, be deleted. There is a path of length at most three between these pairs of vertices. Now, $d(u_xv_p, u_yv_p) \leq \text{diam}(H_1) + 2$ by a path $u_xv_p - u_xv_r - u_{x+1}v_r -$
... $- u_y v_r - u_y v_p$ where $d(u_x v_p, u_x v_r) = d(u_y v_p, u_y v_r) = 1$ and $d(u_x v_r, u_y v_r) \leq \text{diam}(H_1)$. Also, $d(u_x v_q, u_y v_q) \leq \text{diam}(H_1) + 2$ and $d(u_a v_w, u_a v_z) \leq \text{diam}(H_2) + 2$. Thus, the diam(G) remains the same.

Hence, it is clear that at least one graph (say) $H_1$ should be a complete graph and $H_2$ is not a complete graph.

Let $G \cong K_{n_1} \square H_2$ where $n_1 > 2$.

Let the three edges $u_i v_p - u_j v_p$, $u_j v_q - u_x v_q$ and $u_i v_r - u_x v_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There is a path $u_i v_p - u_x v_p - u_j v_p$ of length two between $u_i v_p$ and $u_j v_p$ in $G$. Similarly, $d(u_j v_q, u_x v_q) = d(u_i v_r, u_x v_r) = 2$. Also, the distance between any two other vertices is not affected by the removal of these three edges. Thus, the diam(G) remains the same.

Hence, it follows that $n_1 \leq 2$. Now, we will consider the different cases depending on the value of $n_2$.

**Case 1:** $G \cong K_2 \square H_2$ where $H_2$ is a not complete graph with $n_2 \geq 5$. 
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Suppose that $\text{diam}(H_2) \geq 4$.

Consider a pair of diametral vertices $v_w$ to $v_z$ in $H_2$ where $v_l$ is a vertex in a diametral path between them and is not adjacent to both $v_w$ and $v_z$. Let the three edges $u_1v_w - u_2v_w$, $u_1v_l - u_2v_l$ and $u_1v_z - u_2v_z$, be deleted. There is a path of length three between these pairs of vertices. Consider the vertex $u_1v_w$ in $G$. Then $u_2v_z$, a diametral vertex of $u_1v_w$ is at a distance $\text{diam}(G)$ by a path $u_1v_w - u_1v_{w+1} - \cdots - u_1v_{l-1} - u_2v_{l-1} - u_2v_l - \cdots - u_2v_z$. Thus, the $\text{diam}(G)$ remains the same.

Suppose that $\text{diam}(H_2) = 3$.

Consider a pair of diametral vertices $v_w$ to $v_z$ in $H_2$ where $v_b$ is a vertex not in any of the diametral path between them in $H_2$. Let the three edges $u_1v_w - u_2v_w$, $u_1v_z - u_2v_z$ and $u_1v_b - u_2v_b$, be deleted. There is a path of length at most four between these pairs of vertices. Thus, the $\text{diam}(G)$ remains the same.

Suppose that $\text{diam}(H_2) = 2$.

Suppose that $H_2$ has a universal vertex $v_p$.

Let the three edges $u_1v_q - u_2v_q$, $u_1v_r - u_2v_r$ and $u_1v_l - u_2v_l$ where $q, r, l \neq p$, be deleted. There is a path of length at most three between these pairs of vertices. Thus, the $\text{diam}(G)$ remains
Suppose that $H_2$ does not have a universal vertex and $d(v_w, v_z) = 2$ in $H_2$.

Let the three edges $u_1v_w - u_2v_w$, $u_1v_z - u_2v_z$, and $u_1v_p - u_1v_q$, be deleted. There is a path of length three between these pairs of vertices in $G$. Thus, the distance between any two other vertices is at most three.

**Case 2:** $G \cong K_2 \Box K_{n_2}$ where $n_2 \geq 5$.

Let the three edges $u_1v_2 - u_1v_3$, $u_1v_2 - u_1v_4$, and $u_1v_2 - u_1v_5$, be deleted. There are paths of length two between these pairs of vertices. Thus, the diam($G$) remains the same.

Thus, there exist at least three edges in $G$ that can be deleted without an increase in the diam($G$). Hence, it follows that $n_2 \leq 4$. Now, by an exhaustive verification of all graphs $H_2$ with $n_2 \leq 4$, it follows that $G \cong K_2 \Box K_3$, $K_2 \Box P_3$ and $K_2 \Box P_4$. \qed

**Theorem 2.1.3.** Let $G \cong H_1 \Box H_2$. Then $D^1(G) = 1$ if and only if $H_1$ is a complete graph and either $H_2$ has at least one pair of
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vertices with exactly one diametral path $P$ and no path of length $\text{diam}(H_2) + 1$ which is edge disjoint with $P$ or there exist an edge in $H_2$ that is on all paths of length $\text{diam}(H_2)$, $\text{diam}(H_2) + 1$ between any two diametral vertices in $H_2$.

Proof. Let $u_x, u_y$ be a pair of diametral vertices in $H_1$, by a path $u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y$ and $v_w, v_z$ be a pair of diametral vertices in $H_2$, by a path $v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z$.

Suppose that $H_1$ is a complete graph. If $H_2$ has a pair of vertices $v_w, v_z$, with one diametral path $P$ and no path of length $\text{diam}(H_2) + 1$ edge disjoint with $P$, then $v_p - v_q$ be an edge whose deletion increases the $\text{diam}(H_2)$. If $H_2$ has a pair of vertices $v_w, v_z$, with paths of length $\text{diam}(H_2)$, $\text{diam}(H_2) + 1$ which are not edge disjoint with each other, then $v_p - v_q$ is a common edge in all these paths. Consider a pair of vertices $u_i v_w, u_i v_z$ in $G$. Let an edge $u_i v_p - u_i v_q$, be deleted from the path $u_i v_w - u_i v_{w+1} \ldots u_i v_z$ in $G$, then the $\text{diam}(G)$ increases by a path $u_i v_w - u_j v_w - u_j v_{w+1} - u_j v_{w+2} \ldots u_j v_z - u_i v_z$ where $d(u_j v_w, u_j v_z) = \text{diam}(H_2)$, $d(u_i v_w, u_j v_w) = d(u_i v_z, u_j v_z) = 1$. Also, $d(u_i v_r, u_i v_s) \leq \text{diam}(G)$ where $r, s \in \{1, 2, \ldots, n\}$. The
distance between any two other vertices is not affected by the removal of this edge.

Conversely suppose that $D^1(G) = 1$. If both $H_1$ and $H_2$ are not complete graphs, then at least two edges should be deleted to increase the diam$(G)$.

If $H_1$ and $H_2$ are complete graphs with $n_1, n_2 > 2$, there exist two internally vertex disjoint paths of length two between two non adjacent vertices $u_iv_p$ and $u_jv_q$ in $G$. Thus, at least two edges should be deleted to increase the diam$(G)$.

Hence, it is clear that at least one graph (say) $H_1$ should be a complete graph and $H_2$ is a not complete graph.

Suppose that $d(u_iv_w, u_iv_z) = \text{diam}(H_2)$. Let an edge $u_iv_p - u_iv_q$, be deleted.

If $H_2$ contains two internally edge disjoint paths, one of length diam$(H_2)$ and the other of length diam$(H_2) + 1$ or two internally edge disjoint paths of length diam$(H_2)$ between $v_w$ and $v_z$ in $H_2$, then the diam$(G)$ remains the same, since in both the cases there exist an alternate path of length diam$(H_2) + 1$ or diam$(H_2)$ between $u_iv_w$ and $u_iv_z$ in $G$. 
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If $H_2$ has paths of length $\text{diam}(H_2)$ and $\text{diam}(H_2)+1$ between $v_w$ and $v_z$ in $H_2$, such that all these paths have some edges in common then, the $\text{diam}(G)$ remains the same. Since, all these paths does not have a common edge (say) $v_p - v_q$, even if a delete an edge there exist an alternative path of length $\text{diam}(H_2)+1$ or $\text{diam}(H_2)$ between $u_i v_w$ and $u_i v_z$, without affecting the $\text{diam}(G)$.

Hence, either $H_2$ has at least one pair of vertices with only one diametral path $P$ and no path of length $\text{diam}(H_2)+1$ which is edge disjoint with $P$ or there exist an edge in $H_2$ that is on all paths of length $\text{diam}(H_2)$, $\text{diam}(H_2)+1$ between any two diametral vertices in $H_2$. \hfill \Box

**Corollary 2.1.4.** $G \cong H_1 \square H_2$ is diameter minimal if and only if $H_1 = H_2 = K_2$.

**Proof.** If $G = C_4$, then $G$ is diameter minimal.

Conversely suppose that $G$ is diameter minimal. In Theorem 2.1.3 we have characterized the Cartesian product of graphs whose diameter increases by the deletion of a single edge. Hence, we need to prove the theorem only for such $Gs$.

Let $n_1 > 2$ and $n_2 \geq 2$. 
Let an edge $u_iv_p - u_jv_p$ where $i, j \in \{1, 2, \ldots, n_1\}$ and $p \in \{1, 2, \ldots, n_2\}$, be deleted. There is a path of length two between $u_iv_p$ and $u_jv_p$ in $G$ and the distance between any two other vertices is not affected by the removal of this edge. Thus, the $\text{diam}(G)$ remains the same. Therefore, $n_1 = 2$.

Let $n_1 = 2$ and $n_2 > 2$.

Suppose that $d(v_w, v_z) = \text{diam}(H_2)$. Let an edge $u_1v_z - u_2v_z$, be deleted. Then $d(u_1v_z, u_2v_z) = 3 \leq \text{diam}(G)$ and the distance between $u_1v_w, u_2v_z$ is $\text{diam}(G)$. Also, the distance between any two other vertices is not affected by the removal of this edge. Thus, the $\text{diam}(G)$ remains the same. Hence, for a connected graph $H_2$ with $n_2 > 2$ vertices there exist some $e \in E(G)$ such that $\text{diam}(G - e) < \text{diam}(G)$. Therefore, $n_2 = 2$.

Hence, $H_1 = H_2 = K_2$. \hfill \square

**Theorem 2.1.5.** Let $G \cong H_1\square H_2$.

(a) If both $H_1$ and $H_2$ are complete graphs with $n_1, n_2 > 2$, then $D^1(G) = 2$.

(b) If $H_1$ is a complete graph and $H_2$ is a not complete graph, then $D^1(G) \leq \delta(H_2)$. 
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(c) If both $H_1$ and $H_2$ are not complete graphs, then

\[ D^1(G) \leq \Delta(G) - 1. \]

Proof. Let $u_x, u_y$ be a pair of diametral vertices in $H_1$, by a path $u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y$ and $v_w, v_z$ be a pair of diametral vertices in $H_2$, by a path $v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z$.

(a) $H_1$ and $H_2$ are complete graphs with $n_1, n_2 > 2$.

Let the two edges $u_i v_p - u_j v_p$ and $u_i v_q - u_j v_q$ where $i \neq j \in \{1, 2, \ldots, n_1\}$ and $p \neq q \in \{1, 2, \ldots, n_2\}$, be deleted. Then $d(u_i v_p, u_j v_q) = 3$ by a path $u_i v_p - u_i v_q - u_x v_q - u_j v_q$. Hence, $D^1(G) = 2$.

(b) $H_1$ is a complete graph and $H_2$ is a not complete graph.

Let $d(v_w, v_z) = \text{diam}(H_2)$. Consider a pair of vertices $u_i v_w, u_i v_z$ in $G$. Let the $\delta(H_2)$ edges $u_i v_q - u_i v_r$, where $v_r$s are the neighbours of $v_q$ and $r \in \{1, 2, \ldots, n_2\}$, be deleted. Then, the diam(G) increases by a path $u_i v_w - u_j v_w - u_j v_{w+1} - u_j v_{w+2} - \ldots - u_j v_z - u_i v_z$ where $d(u_i v_w, u_j v_w) = 1, d(u_i v_z, u_j v_z) = 1$ and $d(u_j v_w, u_j v_z) = \text{diam}(H_2)$. Also, the distance between any two other vertices
is not affected by the removal of these edges.
Hence, \( D^1(G) \leq \delta(H_2) \), since \( \deg(v_q) = \delta(H_2) \).

(c) \( H_1 \) and \( H_2 \) are not complete graphs.

Consider a pair of diametral vertices \( u_xv_w, u_yv_z \) in \( G \). Let the edges \( u_yv_{z-1} - u_zv_q \) where \( i \in \{1, 2, ..., n_1\} \), \( v_{z-1} \) is a neighbour of \( v_z \) in \( H_2 \) and \( q \neq z \in \{1, 2, ..., n_2\} \), be deleted. Then the \( \text{diam}(G) \) increases by a path \( u_xv_{w-1} - u_{x+1}v_{w-1} - ... - u_yv_{w-1} - u_yv_{w+1} - ... - u_yv_z - u_yv_{z-1} \) where \( d(u_xv_w, u_xv_z) = \text{diam}(H_2) \) and \( d(u_xv_z, u_yv_{z-1}) = \text{diam}(H_1) - 1 \).
Hence, \( D^1(G) \leq \Delta(G) - 1 \), since \( \deg(u_yv_{z-1}) \leq \Delta(G) \). \( \square \)

**Theorem 2.1.6.** Let \( G \cong H_1 \square H_2 \). Then \( D^{-1}(G) = 1 \) if and only if \( G \) is any one of the following graphs where,

(a) \( H_1 \) is a complete graph and \( H_2 \) is a not complete graph with \( D^{-2}(H_2) = 1 \).

(b) \( H_1 \) is a not complete graph with a universal vertex or there exist a vertex in \( H_1 \) that is on at least one path between any two diametral vertices and \( H_2 \) is a not complete graph with \( D^{-1}(H_2) = 1 \).

**Proof.** Let \( u_x, u_y \) be a pair of diametral vertices in \( H_1 \), by
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a path \( u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y \) and \( v_w, v_z \) be a pair of diametral vertices in \( H_2 \), by a path \( v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z \).

(a) Let \( H_1 \) be a complete graph and \( H_2 \) be a not complete graph with \( D^{-2}(H_2) = 1 \) and the addition of an edge \( v_p - v_q \) in \( H_2 \) decreases the \( \text{diam}(H_2) \) by two. Now, the addition of an edge \( u_1v_p - u_1v_q \) in \( G \) decreases the \( \text{diam}(G) \).

(b) Let \( H_2 \) be a not complete graph with \( D^{-1}(H_2) = 1 \) and \( H_1 \) has a universal vertex \( u_i \) or there exist a vertex \( u_j \) in \( H_1 \) that is on at least one path between any two diametral vertices. Now, the addition of an edge \( u_i v_p - u_i v_q \) or \( u_j v_p - u_j v_q \) in \( G \) decreases the \( \text{diam}(G) \).

Conversely suppose that \( D^{-1}(G) = 1 \).

If both \( H_1 \) and \( H_2 \) are complete graphs, then \( \text{diam}(G) = 2 \) and the addition of an edge in \( G \) will not decrease the \( \text{diam}(G) \).

Suppose that \( H_1 \) is a complete graph.

Consider a pair of diametral vertices \( u_xv_w, u_yv_z \) in \( G \) and \( u_xv_w - u_i v_w - u_i v_{w+1} - \ldots - u_i v_z - u_yv_z \) is a path between them. Let an edge \( u_i v_p - u_i v_q \), be added in \( G \). Then, \( d(u_xv_w, u_i v_w) = 1 \) and
$d(u_iv_z, u_yv_z) = 1$, since $H_1$ is a complete graph. Now, consider the distance between the remaining vertices in the diametral path, then the $\text{diam}(G)$ decreases by one, only if $d(u_iv_w, u_iv_z) = \text{diam}(H_2) - 2$. Hence, to decrease the $\text{diam}(G)$ by one, the distance between $u_iv_w$ and $u_iv_z$ should be decreased by two, by the addition of a single edge. Thus, $H_2$ is a not complete graph with $D^{-2}(H_2) = 1$.

Suppose that $D^{-1}(H_2) = 1$.

Consider a pair of diametral vertices $u_xv_w, u_yv_z$ in $G$. Let an edge $u_iv_p - u_iv_q$, be added in $G$. If $u_i$ is not a universal vertex of $H_1$, then a diametral path between them does not contain the edge $u_iv_p - u_iv_q$. Thus, the $\text{diam}(G)$ remains the same. Hence, $H_1$ is a not complete graph with a universal vertex.

Let $u_x$, $u_y$ and $u_s$, $u_t$ be the pairs of diametral vertices of $H_1$ where $u_i$ is a vertex in a diametral path between $u_x$, $u_y$ and $u_i$ is a vertex not in any of the diametral path between $u_s$, $u_t$ in $H_1$. Consider the pairs of diametral vertices $u_xv_w, u_yv_z$ and $u_sv_w, u_tv_z$ in $G$. Let an edge $u_iv_p - u_iv_q$, be added in $G$. Then, $d(u_xv_w, u_yv_z) = \text{diam}(G) - 1$, by a path $u_xv_w - u_{x+1}v_w - \ldots - u_iv_w - u_{i+1}v_w - \ldots - u_{i+1}v_z - \ldots - u_yv_z$. Also,
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\[ d(u_s v_w, u_t v_z) = \text{diam}(G), \] since \( u_i \) is not in any of the diametral path between \( u_s \) and \( u_t \) in \( H_1 \). Thus, the \( \text{diam}(G) \) remains the same. Hence, \( H_1 \) is a not complete graph with a universal vertex or there exist a vertex in \( H_1 \) that is on at least one path between any two diametral vertices.

**Corollary 2.1.7.** There does not exist a graph \( G \cong H_1 \square H_2 \) such that \( G \) is diameter maximal.

**Proof.** In Theorem 2.1.6 we have characterized the Cartesian product of graphs whose diameter decreases by the addition of a single edge. Hence, we need to prove the theorem only for such \( G \)s.

Let \( d(u_x, u_y) = \text{diam}(H_1) \) and \( d(v_w, v_z) = \text{diam}(H_2) \). Consider a pair of diametral vertices \( u_x v_w, u_y v_z \) in \( G \). Let an edge \( u_x v_{w+1} - u_{x+1} v_w \) where \( u_{x+1} \) is a neighbour of \( u_x \) in \( H_1 \) and \( v_{w+1} \) is a neighbour of \( v_w \) in \( H_2 \), be added in \( G \). Then the added edge does not decrease the distance between them in \( G \). Thus, \( d(u_x v_p, u_y v_q) = \text{diam}(G) \). Hence, there exist \( e \not\in E(G) \) such that \( \text{diam}(G + e) = \text{diam}(G) \).  

□
2.2 Diameter variability of the strong product of graphs

If both $H_1$ and $H_2$ are complete graphs, then $G \cong H_1 \boxtimes H_2$ is a complete graph and the deletion of any edge increases the diam(G).

**Theorem 2.2.1.** Let $G \cong H_1 \boxtimes H_2$. Then $D^0(G) \geq 6$.

**Proof.** Let $G \cong H_1 \boxtimes H_2$.

Then $\text{diam}(G) = \max\{\text{diam}(H_1), \text{diam}(H_2)\}$.

We shall prove the theorem by showing that there exist at least six edges in $G$ that can be deleted without altering the diam(G) by considering the following cases.

Let $u_x, u_y$ be a pair of diametral vertices in $H_1$, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$.

**Case 1:** $H_1$ is a not complete graph and $H_2$ is any connected graph with $n_2 \geq 4$ and $\text{diam}(H_2) < \text{diam}(H_1)$.

We shall prove that $D^0(G) \geq n_1 m_2$.

Let $d(v_w, v_z) = L$ in $H_2$ by a path $v_w - v_{w+1} - v_{w+2} - ... -$
2.2. Diameter variability of the strong product of graphs

$v_{z-1} - v_z$. Consider a pair of diametral vertices $u_x v_w$, $u_y v_z$ in $G$. Let the edges $u_i v_p - u_i v_q$ where $i \in \{1, 2, \ldots, n_1\}$ and $p, q \in \{1, 2, \ldots, n_2\}$, be deleted. There are paths $u_i v_p - u_{i+1} v_{p+1} - u_i v_{p+2} - \ldots - u_i v_{q-1} - u_{i+1} v_{q} - u_i v_q$ or $u_i v_p - u_{i+1} v_{p+1} - u_i v_{p+2} - \ldots - u_{i+1} v_{q-1} - u_i v_q$ of length $\text{diam}(H_2) + 1$ or $\text{diam}(H_2)$ between $u_i v_p$ and $u_j v_p$ when $\text{diam}(H_2)$ is odd or even respectively, where $u_{i+1}$ is a neighbour of $u_i$ in $H_1$. Also, $d(u_x v_w, u_y v_z) = \text{diam}(G)$ by a path $u_x v_w - u_{x+1} v_w - u_{x+2} v_w - \ldots - u_i v_w - \ldots - u_{y-2} v_{z-2} - u_{y-1} v_{z-1} - u_y v_z$ where $d(u_x v_w, u_i v_w) = \text{diam}(H_1) - L$, and $d(u_i v_w, u_y v_z) = L$. Thus, the $\text{diam}(G)$ remains the same.

Now, we consider $n_2 = 2, 3$.

(a) $G \cong P_3 \boxtimes K_2$.

Let the bold edges in Fig 2.2 be deleted. Then it is clear that $D^0(G) = 6$.

Fig 2.2: $P_3 \boxtimes K_2$.

(b) $H_1$ is a not complete graph with $n_1 \geq 4$ and $H_2 = K_2$. 

}\end{document}
Consider the three vertices \( u_p, u_q \) and \( u_r \) in \( H_1 \) which form a path \( P_3 \). Now, \( P_3 \boxtimes K_2 \) is a subgraph of \( G \). Let the six bold edges as in Fig 2.2 and an edge \( u_1 v_s - u_2 v_s \), be deleted. There is a path of length two between these pairs of vertices and the distance between any two other vertices is not affected by the removal of these edges. Thus, \( D^0(G) > 6 \).

(c) \( G \cong P_3 \boxtimes K_3, P_4 \boxtimes P_3 \) and \( P_4 \boxtimes K_3 \).

\( \text{Fig 2.3: (i) } P_3 \boxtimes K_3 \) (ii) \( P_4 \boxtimes P_3 \) (iii) \( P_4 \boxtimes K_3 \).

From Fig 2.3 it is clear that \( D^0(G) > 6 \).

(d) \( H_1 \) is a not complete graph with \( n_1 \geq 4 \) and \( n_2 = 3 \).

Let the edges \( u_i v_1 - u_j v_1 \) and \( u_i v_3 - u_j v_3 \) where \( i, j \in \{1, \ldots, n_1\} \), be deleted. There is a path of length three between these pairs of vertices. Also, \( d(u_x v_1, u_y v_1) \leq \text{diam}(H_1) \) by a path \( u_x v_1 - u_{x+1} v_2 - \ldots - u_{y-1} v_2 - u_y v_1 \). Also, \( d(u_i v_3, u_j v_3) \leq \text{diam}(H_1) \).
2.2. Diameter variability of the strong product of graphs

Thus, $D^0(G) > 6$.

**Case 2:** $H_1$ and $H_2$ are connected not complete graphs with $n_1, n_2 \geq 4$ and $\text{diam}(H_1) = \text{diam}(H_2)$.

Let $G \cong P_4 \boxtimes P_4$. Then clearly $D^0(G) > 6$.
Consider $G \cong H_1 \boxtimes H_2$. We shall prove that $D^0(G) \geq m_1 + m_2$.

Suppose that $u_x, u_y$ and $v_w, v_z$ are the pairs of diametral vertices in $H_1$ and $H_2$ respectively. Let the edges $u_1v_p - u_1v_q, u_iv_1 - u_jv_1$ where $p, q \in \{1, 2, \ldots, n_2\}$ and $i, j \in \{1, 2, \ldots, n_1\}$, be deleted.
Then, $d(u_1v_w, u_1v_z) = \text{diam}(H_2)$ by a path $u_1v_w - u_2v_{w+1} - u_2v_{w+2} - \ldots - u_2v_{z-1} - u_1v_z$ and $d(u_xv_1, u_yv_1) = \text{diam}(H_1)$ by a path $u_xv_1 - u_{x+1}v_2 - u_{x+2}v_2 - \ldots u_{y-1}v_2 - u_yv_1$. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, the diam(G) remains the same and hence $D^0(G) > 6$.

Next, we consider $n_1 \geq 3$ and $n_2 = 3$.

(a) $G \cong P_3 \boxtimes P_3$.

Let the bold edges in Fig 2.4 be deleted. Then it is clear that $D^0(G) > 6$.

(b) $G \cong H_1 \boxtimes P_3$ where $n_1 = 4$. 

Fig 2.4: $P_3 \boxtimes P_3$.

By an exhaustive verification of all such graphs, it follows that $D^0(G) > 6$.

(c) $G \cong H_1 \boxtimes P_3$ where $n_1 \geq 5$.

We shall prove that $D^0(G) \geq 2m_1$.

Let the edges $u_p v_1 - u_q v_1$ and $u_p v_3 - u_q v_3$ where $p, q \in \{1, 2, \ldots, n_1\}$, be deleted. Then, $d(u_x v_1, u_y v_1) \leq \text{diam}(H_1)$ by a path $u_x v_1 - u_{x+1} v_2 - \ldots - u_{y-1} v_2 - u_y v_1$ and $d(u_x v_3, u_y v_3) \leq \text{diam}(H_1)$. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, the diam(G) remains the same and hence $D^0(G) > 6$.

Corollary 2.2.2. $D^0(G) = 6$ if and only if $H_1 = P_3$ and $H_2 = K_2$.

Corollary 2.2.3. Let $G \cong H_1 \boxtimes H_2$ where $H_1$ and $H_2$ are connected graphs with $\text{diam}(H_2) < \text{diam}(H_1)$. Then $D^0(G) \geq n_1 m_2$. 
2.2. Diameter variability of the strong product of graphs

Theorem 2.2.4. Let \( G \cong H_1 \boxtimes H_2 \). Then \( D^1(G) = 1 \) if and only if \( G \) is any one of the following graphs where,

(a) both \( H_1 \) and \( H_2 \) are complete graphs.
(b) \( H_1 \) and \( H_2 \) are not complete graphs with \( \text{diam}(H_1) = \text{diam}(H_2) \)
and either \( H_1 \) or \( H_2 \) have at least one pair of vertices with exactly one diametral path or there exist an edge in \( H_1 \) or \( H_2 \) that is on all diametral paths between any two vertices.

Proof. Let \( G \cong K_{n_1} \boxtimes K_{n_2} \) where \( n_1, n_2 \geq 2 \). Then \( G \) is a complete graph and the deletion of any edge increases the \( \text{diam}(G) \).

Let \( H_1 \) and \( H_2 \) are not complete graphs with \( \text{diam}(H_1) = \text{diam}(H_2) \) and either \( H_1 \) or \( H_2 \) have at least one pair of vertices with exactly one diametral path or there exist an edge in \( H_1 \) or \( H_2 \) that is on all diametral paths between any two vertices.

Let \( u_x, u_y \) be a pair of diametral vertices in \( H_1 \), by a path \( u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y \) and \( v_w, v_z \) be a pair of diametral vertices in \( H_2 \), by a path \( v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z \).

Consider a pair of diametral vertices \( u_xv_w, u_yv_z \) in \( G \), by a path \( u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2} - \ldots - u_{y-1}v_{z-1} - u_yv_z \). Let an edge \( u_xv_w - u_{x+1}v_{w+1} \) be deleted. Then, \( d(u_xv_w, u_yv_z) = \text{diam}(G) + 1 \).
by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_{w+1} \ldots u_{y-1}v_{z-1} - u_yv_z$ where 
\[ d(u_xv_w, u_{x+1}v_{w+1}) = 2, \quad d(u_{x+1}v_{w+1}, u_yv_z) = \text{diam}(G) - 1. \]

Conversely suppose that $D^1(G) = 1$.

Suppose that $H_1$ is a not complete graph and $H_2$ is a complete graph.

Let an edge $u_iv_p - u_iu_q$ or $u_jv_p - u_jv_q$ or $u_iv_p - u_jv_{p+1}$, be deleted. Then $d(u_iv_p, u_iu_q) = d(u_jv_p, u_jv_q) = d(u_iv_p, u_jv_{p+1}) = 2$ by the paths $u_iv_p - u_{i+1}v_q - u_iv_q$, $u_jv_p - u_{j+1}v_{p+1} - u_jv_{p+1}$ and $u_iv_p - u_iu_{p+1} - u_jv_{p+1}$ respectively. Also, the distance between any two other vertices is not affected by the removal of this edge. Thus, when one factor is a complete graph and the other factor is a not complete graph, a minimum of two edges should be deleted to increase the diam$(G)$. Hence, both the factors should be complete. This proves (a).

Suppose that $H_1$ and $H_2$ are not complete graphs with 
\[ \text{diam}(H_1) > \text{diam}(H_2). \]

Consider a pair of diametral vertices $u_xv_w, u_yv_z$ in $G$ by a path $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2} \ldots u_{y-1}v_{z-1} - u_yv_z$. Let an edge $u_xv_w - u_{x+1}v_{w+1}$, be deleted. Then, $d(u_xv_w, u_yv_z) = \text{diam}(H_2) + 1$.
by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_{w+1} \ldots u_{y-1}v_{z-1} - u_yv_z$ where $d(u_xv_w, u_{x+1}v_{w+1}) = 2$, $d(u_{x+1}v_{w+1}, u_yv_z) = \text{diam}(H_2) - 1$.

Hence, $\text{diam}(G)$ remains the same. Thus, when $H_1$ and $H_2$ are not complete graphs with different diameter, at least two edges should be deleted to increase the $\text{diam}(G)$.

Suppose that $H_1$ and $H_2$ are not complete graphs with $\text{diam}(H_1) = \text{diam}(H_2)$.

Consider a pair of diametral vertices $u_xv_w, u_yv_z$ in $G$. Since, $\text{diam}(H_1) = \text{diam}(H_2)$, $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2} \ldots u_{y-1}v_{z-1} - u_yv_z$ is a shortest path between them in $G$. Then, the deletion of an edge $u_iv_j - u_{i+1}v_{j+1}$ from this path increases the $\text{diam}(G)$ only if either there exist only one diametral path between $u_x$, $u_y$ in $H_1$ and $v_w, v_z$ in $H_2$ or $u_i - u_{i+1}$ is an edge in $H_1$ that is on all diametral paths between any two vertices in $H_1$ and $v_j - v_{j+1}$ is an edge in $H_2$ that is on all diametral paths between any two vertices in $H_2$. Otherwise, there exist an alternative path of length $\text{diam}(H_1)$ between $u_xv_w, u_yv_z$ in $G$. Hence, $H_1$ and $H_2$ are not complete graphs with $\text{diam}(H_1) = \text{diam}(H_2)$ and either $H_1$ or $H_2$ have at least one pair of vertices with exactly one diametral path or there exist an edge in $H_1$ or $H_2$ that is on all diametral paths between any two vertices. This proves (b).
Corollary 2.2.5. \( G \cong H_1 \boxtimes H_2 \) is diameter minimal if and only if both \( H_1 \) and \( H_2 \) are complete graphs.

Theorem 2.2.6. Let \( G \cong H_1 \boxtimes H_2 \).

Then \( D^1(G) \leq \alpha(1 + \delta(H_2)) \) where \( \alpha \) is the minimum number of edge disjoint paths of length \( \text{diam}(H_1) \) between any two vertices in \( H_1 \).

Proof. Let \( u_x \) and \( u_y \) be a pair of diametral vertices in \( H_1 \), by a path \( u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y \). Consider a pair of diametral vertices \( u_x v_z \) and \( u_y v_z \) in \( G \). Let the edges \( u_x v_z - u_q v_z, u_x v_z - u_q v_r \) where \( u_q \)s are the vertices adjacent to \( u_x \) in \( H_1 \) and \( v_r \)s are the vertices adjacent to \( v_z \) in \( H_2 \), be deleted. Then, \( d(u_x v_z, u_y v_z) = \text{diam}(G) + 1 \) by a path \( u_x v_z - u_{x+1} v_z - u_{x+1} v_z - ... - u_{y-1} v_z - u_y v_z \) where \( d(u_{x+1} v_z, u_y v_z) = \text{diam}(G) - 1, d(u_x v_z, u_{x+1} v_z) = 2 \).

Also, \( d(u_x v_z, u_q v_z) = 2 \) and \( d(u_x v_z, u_q v_r) = 2 \), since there are paths of length two between them.

Thus, \( D^1(G) \leq \alpha(1 + \delta(H_2)) \).

Theorem 2.2.7. Let \( G \cong H_1 \boxtimes H_2 \) be connected graph. Then \( D^{-1}(G) = 1 \) if and only if \( H_2 \) has a universal vertex and \( H_1 \) is a
connected graph with \( \text{diam}(H_1) \geq 4 \) and \( D^{-2}(H_1) = 1 \) when an edge is added between a diametral vertex and any other vertex of \( H_1 \) and \( D^{-1}(H_1) = 1 \) when an edge is added between any two other vertices of \( H_1 \).

**Proof.** Let \( G \cong H_1 \Box H_2 \) and \( \text{diam}(G) = \text{diam}(H_1) \).

Let \( u_x, u_y \) be a pair of diametral vertices in \( H_1 \), by a path \( u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y \) and \( v_w, v_z \) be a pair of diametral vertices in \( H_2 \), by a path \( v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z \). Suppose that \( v_1 \) is a universal vertex of \( H_2 \).

Let \( D^{-1}(H_1) = 1 \) where \( \text{diam}(H_1) \geq 4 \).

Consider a pair of diametral vertices \( u_x v_w, u_y v_z \) in \( G \). Let an edge \( u_p v_1 - u_q v_1 \) where \( u_p \neq u_x, u_q \neq u_y \), be added in \( G \). Then, \( d(u_x v_w, u_y v_z) = \text{diam}(G) - 1 \) by a path \( u_x v_w - u_{x+1} v_1 - u_{x+2} v_1 - \ldots - u_{y-1} v_1 - u_y v_z \) where \( d(u_x v_w, u_{x+1} v_1) = 1 \), \( d(u_{x+1} v_1, u_{y-1} v_1) = \text{diam}(G) - 3 \) and \( d(u_{y-1} v_1, u_y v_z) = 1 \).

Consider a pair of diametral vertices \( u_x v_w, u_y v_z \) in \( G \). Let an edge \( u_x v_1 - u_y v_1 \), be added in \( G \). Then, \( d(u_x v_w, u_y v_z) = 3 \) by a path \( u_x v_w - u_x v_1 - u_y v_1 - u_y v_z \).
Suppose that $D^{-2}(H_1) = 1$ where $\text{diam}(H_1) \geq 4$.

Consider a pair of diametral vertices $u_xv_w, u_yv_z$ in $G$. Let an edge $u_xv_1 - u_iv_1$ where $u_i$ is a vertex in a diametral path between $u_x$ and $u_y$ in $H_1$, be added in $G$. Then, $d(u_xv_w, u_yv_z) = \text{diam}(G) - 1$ by a path $u_xv_w - u_xv_1 - u_iv_1 - ... - u_{y-1}v_1 - u_yv_z$ where

\begin{align*}
d(u_xv_w, u_xv_1) &= 1, \\
d(u_xv_1, u_{y-1}v_1) &= \text{diam}(G) - 3 \quad \text{and} \\
d(u_{y-1}v_1, u_yv_z) &= 1. \end{align*}

Thus, the distance between any two vertices in $G$ is at most diam(G)-1.

Conversely suppose that $D^{-1}(G) = 1$. If both $H_1$ and $H_2$ are complete graphs, then $G$ is a complete graph. If $\text{diam}(H_1) = 2$, then the addition of a single edge in $G$ will not make $G$ a complete graph. Also, if $\text{diam}(H_1) = 3$, then the addition of a single edge in $G$ will not decrease the $\text{diam}(G)$, since there exist a path of length at least three between any pair of diametral vertices in $G$. Thus, it is clear that $H_1$ is a connected graph with $\text{diam}(H_1) \geq 4$.

Suppose that $H_1$ is any connected graph and $H_2$ is any connected graph without a universal vertex.

Let $v_p$ and $v_q$ be a pair of non adjacent vertices in $H_2$. Con-
2.2. Diameter variability of the strong product of graphs

Consider a pair of diametral vertices $u_xv_q$, $u_yv_q$ in $G$. Let an edge $u_iv_p - u_jv_p$, be added in $G$. Since $v_p$ is not adjacent to $v_q$, the diametral path between $u_xv_q$ and $u_yv_q$ does not contain the edge $u_iv_p - u_jv_p$ in $G$. Hence, to decrease the diam$(G)$, $H_2$ should contain a universal vertex.

Suppose that $H_2$ has a universal vertex $v_1$. Consider a pair of diametral vertices $u_xv_w$, $u_yv_w$ in $G$. Let an edge $u_iv_1 - u_jv_1$, be added in $G$.

Let $i \neq x, j \neq y$.

Consider a diametral path $u_xv_w - u_{x+1}v_1 - u_{x+2}v_1 - \ldots - u_{y-1}v_1 - u_yv_w$ between $u_xv_w$, $u_yv_w$ in $G$. Then $d(u_xv_w, u_{x+1}v_1) = 1$ and $d(u_{y-1}v_1, u_yv_w) = 1$, since $H_2$ has a universal vertex. Now, consider the distance between the remaining vertices in the diametral path. Then, the diam$(G)$ decreases by one only if $d(u_{x+2}v_1, u_{y-1}v_1) = \text{diam}(H_1) - 2 - 1 = \text{diam}(H_1) - 3$. Hence, to decrease the diam$(G)$ by one, the distance between $u_xv_1$ and $u_yv_1$ should be decreased by one, by the addition of a single edge.

Let $i = x, j = y$.

Then, $d(u_xv_w, u_yv_w) = 3$ by a path $u_xv_w - u_xv_1 - u_yv_1 - u_yv_w$, since $H_2$ has a universal vertex. From the previous case it follows
that diam(G) decreases, only if \( d(u_p v_1, u_q v_1) \leq \text{diam}(H_1) - 1 \).

Hence, to decrease the diam(G) by one, the distance between \( u_x v_1 \) and \( u_y v_1 \) should be decreased by one, by the addition of a single edge.

Now, let \( i = x, j \neq y \).

Consider a diametral path \( u_x v_w - u_x v_1 - u_{x+1} v_1 - \ldots - u_{y-1} v_1 - u_y v_w \) between \( u_x v_w, u_y v_w \) in \( G \). Then \( d(u_x v_w, u_x v_1) = 1 \) and \( d(u_{y-1} v_1, u_y v_w) = 1 \), since \( H_2 \) has a universal vertex. Now, consider the distance between the remaining vertices in the diametral path. Then, the diam(G) decreases by one, only if \( d(u_x v_1, u_{y-1} v_1) = [\text{diam}(H_1) - 1] - 2 = \text{diam}(H_1) - 3 \). Hence, to decrease the diam(G) by one, the distance between \( u_x v_1 \) and \( u_{y-1} v_1 \) should be decreased by two, by the addition of a single edge. \( \square \)

**Corollary 2.2.8.** There does not exist a graph \( G \cong H_1 \boxtimes H_2 \) such that \( G \) is diameter maximal.

**Proof.** In Theorem 2.2.7 we have characterized the strong product of graphs whose diameter decreases by the addition of a single edge. Hence, we need to prove the theorem only for such \( Gs \).
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Suppose that $H_2$ is a not complete graph with a universal vertex and $H_1$ is a connected graph with $D^{-1}(H_1) = 1$ or $D^{-2}(H_1) = 1$ with $\text{diam}(H_1) \geq 4$. Let an edge $u_xv_p - u_xv_q$ be added in $G$, then the diam($G$) remains the same, since $\text{diam}(G) = \text{diam}(H_1)$.

Suppose that $H_2$ is a complete graph and $H_1$ is a connected graph with $D^{-1}(H_1) = 1$ or $D^{-2}(H_1) = 1$ with $\text{diam}(H_1) \geq 4$. Let the three vertices $u_x$, $u_s$ and $u_r$ form a $P_3$ in $H_1$. Consider a pair of diametral vertices $u_xv_p, u_yv_p$ in $G$. Let an edge $u_xv_q - u_rv_q$ where $v_q$ is a neighbour of $v_p$ in $H_2$, be added. Then the addition of an edge $u_xv_q - u_rv_p$ does not decrease the distance between them in $G$. Thus, $d(u_xv_p, u_yv_p) = \text{diam}(G)$. Hence, there exist some $e \notin E(G)$ such that $\text{diam}(G + e) = \text{diam}(G)$. \qed

2.3 Diameter variability of the lexicographic product of graphs

If both $H_1$ and $H_2$ are complete graphs, then $G \cong H_1 \circ H_2$ is a complete graph and the deletion of any edge increases the
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diam(G).

**Theorem 2.3.1.** Let $G \cong H_1 \circ H_2$. Then $D^0(G) \geq 3$.

**Proof.** Let $G \cong H_1 \circ H_2$. Then $\text{diam}(G) = \text{diam}(H_1)$.

We prove the theorem by showing that there exist at least three edges in $G$ that can be deleted without altering the $\text{diam}(G)$ by considering the following cases.

Let $u_x, u_y$ be a pair of diametral vertices in $H_1$, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and $v_w, v_z$ be a pair of diametral vertices in $H_2$, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

**Case 1 :** $H_1$ is a complete graph and $H_2$ is a not complete graph or a disconnected graph with $m_2 \geq 1$.

(a) Let $m_2 \geq 2$.

We shall prove that $D^0(G) \geq n_1 m_2$.

Suppose that $G \cong K_{n_1} \circ H_2$, then $\text{diam}(G) = 2$. Let the edges $u_i v_p - u_i v_q$ where $i \in \{1, 2, ... , n_1\}$ and $p, q \in \{1, 2, ... , n_2\}$, be deleted. There are paths $u_i v_p - u_{i+1} v_p - u_i v_q$ of length two between each pair of vertices in $G$. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geq n_1 m_2 \geq 4$. 
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(b) Let $m_2 = 1$.

Suppose that $n_1 = 2$ and $n_2 = 3$.

Let the bold edges in Fig 2.6 be deleted. Then it is clear that $D^0(G) = 3$.

Suppose that $n_1 = 2$ and $n_2 \geq 4$.

Let the edges $u_iv_p - u_jv_q$, $u_iv_q - u_jv_p$, $u_iv_q - u_jv_r$ and $u_iv_r - u_jv_q$ where $v_q$ is adjacent to $v_p$ in $H_2$, be deleted. There are paths $u_iv_p - u_jv_p - u_jv_q$, $u_iv_q - u_jv_p - u_jv_q$, $u_iv_q - u_iv_p - u_jv_r$ and $u_iv_r - u_jv_p - u_jv_q$ of length two between each pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geq 4$.

Suppose that $n_1 = 3$ and $n_2 = 3$.

Let the bold edges in Fig 2.5 be deleted, then it is clear that $D^0(G) > 3$.

Fig 2.5: $G : D^0(G) > 3$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2_5.png}
\caption{Graph $G$ with $D^0(G) > 3$.}
\end{figure}
Suppose that $n_1 > 3$ and $n_2 \geq 3$.
Let the edges $u_i v_1 - u_j v_1$ where $i, j \in \{1, 2, \ldots, n_1\}$, be deleted.
There are paths $u_i v_1 - u_i v_2 - u_j v_1$ of length two between these pairs of vertices in $G$. Also, the distance between any two other vertices is not affected by the removal of these edges.
Thus, $D^0(G) \geq 4$.

**Case 2 :** $H_1$ is a complete graph and $H_2$ is a totally disconnected graph.

(a) Let $n_1 = 2$.
Then $G$ has diameter two and the deletion of any edge increases the diam(G).

(b) Let $n_1 \geq 3$.
Let the edges $u_i v_1 - u_j v_1$, $u_i v_1 - u_j v_2$, $u_i v_2 - u_j v_2$ and $u_i v_2 - u_j v_1$, be deleted. There are paths $u_i v_1 - u_x v_1 - u_j v_1$, $u_i v_1 - u_x v_2 - u_j v_2$, $u_i v_2 - u_x v_2 - u_j v_2$ and $u_i v_2 - u_x v_1 - u_j v_1$ of length two between each pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.
Thus, $D^0(G) \geq 4$.

**Case 3 :** $H_1$ is a not complete graph and $H_2$ is a not
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complete graph or a disconnected graph with \( m_2 \geq 1 \).

(a) Let \( n_1 \geq 4 \).
We shall prove that \( D^0(G) \geq n_1m_2 \).
Let the edges \( u_i v_p - u_i v_q \) where \( i \in \{1, 2, \ldots, n_1\} \) and \( p, q \in \{1, 2, \ldots, n_2\} \), be deleted. There are paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, \( D^0(G) \geq n_1m_2 \geq 4 \).

(b) Let \( n_1 = 3 \).
Let the edges \( u_1 v_i - u_1 v_j, u_1 v_j - u_2 v_j, u_2 v_j - u_3 v_j \) and \( u_3 v_i - u_3 v_j \) where \( u_2 \) is adjacent to \( u_1 \) and \( u_3 \) in \( H_1 \), be deleted. There are paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, \( D^0(G) \geq 4 \).

**Case 4:** \( H_1 \) is a not complete graph and \( H_2 \) is a totally disconnected graph.

(a) \( H_1 \) is a not complete graph with diameter two in which no two adjacent vertices of \( H_1 \) have a path of length two between them.
Then, \( \text{diam}(G) = 2 \) and the deletion of an edge increases the \( \text{diam}(G) \).

(b) \( H_1 \) is a not complete graph with diameter two in which there exist at least one pair of adjacent vertices with a path of length two between them.

Let the edges \( u_iv_1 - u_jv_1, u_iv_1 - u_jv_2, u_iv_2 - u_jv_2 \) and \( u_iv_2 - u_jv_1 \) where there is a path of length two between \( u_i \) and \( u_2 \) in \( H_1 \), be deleted. There are paths \( u_iv_1 - u_xv_1 - u_jv_1, u_iv_1 - u_xv_2 - u_jv_2, u_iv_2 - u_xv_2 - u_jv_2 \) and \( u_iv_2 - u_xv_1 - u_jv_1 \) of length two between each pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.

Thus, \( D^0(G) \geq 4 \).

(c) \( H_1 = P_4 \) and \( n_2 = 2 \).

Let the bold edges in Fig 2.6 be deleted. Then it is clear that \( D^0(G) = 3 \).

(d) \( H_1 = P_4 \) and \( n_2 > 2 \).

Let the edges \( u_iv_1 - u_jv_1 \) and \( u_iv_2 - u_jv_2 \) where \( i, j \in \{1, 2, 3, 4\} \), be deleted. There are paths of length at most three between these pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.
Thus, \( D^0(G) > 4 \).

(e) Let \( \text{diam}(H_1) \geq 3 \).

We shall prove that \( D^0(G) \geq m_1 \).

Let the edges \( u_iv_1 - u_jv_1 \) where \( i, j \in \{1, 2, ..., n_1\} \), be deleted. There are paths of length at most \( \text{diam}(G) \) between these pairs of vertices. Also, \( d(u_xv_1, u_yv_1) = \text{diam}(G) \) by a path \( u_xv_1 - u_{x+1}v_2 - u_{x+2}v_2 ... u_{y-1}v_2 - u_yv_1z \) and the distance between any two other vertices is not affected by the removal of these edges. Thus, \( D^0(G) \geq m_1 \geq 4 \).

Hence, \( D^0(G) \geq 3 \). \( \square \)

**Corollary 2.3.2.** \( D^0(G) = 3 \) if and only if \( G \) is any one of the graphs shown in Fig 2.6.

![Fig 2.6: The graphs G : D^0(G) = 3.](image)

**Corollary 2.3.3.** Let \( G \cong H_1 \circ H_2 \) where \( H_1 \) and \( H_2 \) are connected graphs. Then \( D^0(G) \geq n_1m_2 \).
Theorem 2.3.4. Let $G \cong K_{n_1} \circ H_2$ where $n_1 \geq 3$.
Then $D^0(G) = n_2^2 m_1 + n_1 m_2 - (2n_1 n_2 - 3)$.

Proof. Consider a spanning tree $T$ of diameter three, of $G$ as shown in Fig 2.7. From $T$, let us construct a spanning subgraph $H$ of $G$ having diameter two as follows.

Consider the vertices $u_1v_p, u_x v_q$ where $x \in \{2, 3, ..., n_1\}$ and $p, q \in \{2, 3, ..., n_2\}$. Then, $d(u_1 v_p, u_x v_q) = 3$. Let the edges $u_2 v_1 - u_x v_p$ where $x \in \{3, 4, ..., n_1\}$ and $p \in \{1, 2, ..., n_2\}$, be added in $T$. Now, consider the vertices $u_1 v_p, u_2 v_q$ where $p \in \{2, 3, ..., n_2\}$, then $d(u_1 v_p, u_2 v_q) = 3 > 2$. Let the edges $u_3 v_1 - u_1 v_p$ and $u_3 v_1 - u_2 v_p$ where $p \in \{2, 3, ..., n_2\}$, be added in $T$.

Let the resulting spanning subgraph of $G$ be denoted by $H$. Then $H$ has diameter two.
Hence, $D^0(G) \geq n_2^2 m_1 + n_1 m_2 - (2n_1 n_2 - 3)$.

Now, to prove the reverse inequality, we proceed as follows. From Corollary 2.3.3 it follows that if the $n_1 m_2$ edges $u_i v_p - u_i v_q$ where $i \in \{1, 2, ..., n_1\}$ and $p, q \in \{1, 2, ..., n_2\}$ are deleted, then the diam($G$) remains the same. Let the edges $u_i v_p - u_j v_p$ where
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Fig 2.7: A spanning tree $T$ and the spanning subgraph $H$ of $G$.

Let the edges $u_iv_p - u_jv_q$ where $i, j \in \{1, 2, ..., n_1\}$, $p, q \in \{1, 2, ..., n_2\}$ except $u_1v_1 - u_rv_1$, $u_2v_1 - u_rv_1$ where $r \in \{2, 3, ..., n_2\}$, be deleted. There is a path $u_iv_p - u_xv_1 - u_jv_q$ of length two between each pair of vertices.

Now, let the edges $u_iv_p - u_jv_q$ where $i, j \in \{1, 2, ..., n_1\}$, $p, q \in \{1, 2, ..., n_2\}$ except $u_1v_1 - u_iv_p$, $u_2v_1 - u_jv_p$, $u_3v_1 - u_1v_r$ and $u_3v_1 - u_2v_r$ where $i \in \{2, 3, ..., n_2\}$, $j \in \{1, 3, ..., n_2\}$, $p \in \{2, 3, ..., n_2\}$ and $r \in \{2, 3, ..., n_2\}$, be deleted. There are paths $u_iv_p - u_1v_1 - u_jv_q$, $u_1v_p - u_3v_1 - u_2v_q$ of length two between each pair of vertices. In both the cases the diam$(G)$ remains the same.

Thus we have a spanning subgraph $H$ with diameter two as shown in Fig 2.7 and the deletion of any edge from $H$ increases the diam$(H)$. So, $D^0(G) \leq n_2^2m_1 + n_1m_2 - (2n_1n_2 - 3)$. 
Hence, \( D^0(G) = n_2^2m_1 + n_1m_2 - (2n_1n_2 - 3) \). \( \square \)

**Theorem 2.3.5.** Let \( G \cong H_1 \circ H_2 \) where \( H_1 \) and \( H_2 \) are connected graphs with \( \text{diam}(H_2) < \text{diam}(H_1) \). Then
\[
D^0(G) \geq n_2^2m_1 - (m_1n_2 + 2m_1m_2).
\]

**Proof.** Let \( u_x, u_y \) be a pair of diametral vertices in \( H_1 \), by a path \( u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y \).

Suppose that \( d(v_p, v_q) = L \) in \( H_2 \) by a path \( v_p - v_{p+1} - v_{p+2} - \ldots - v_{q-1} - v_q \). Consider a pair of diametral vertices \( u_xv_p, u_yv_q \) in \( G \). Let the \( n_1m_2 \) edges \( u_iv_p - u_iv_q \) where \( i \in \{1, 2, \ldots, n_1\} \) and \( p, q \in \{1, 2, \ldots, n_2\} \), be deleted. Then from Corollary 2.3.3 it follows that the \( \text{diam}(G) \) remains the same. Now, let the \( n_2^2m_1 - (m_1n_2 + 2m_1m_2) \) edges \( u_iv_p - u_jv_q \) where \( i, j \in \{1, 2, \ldots, n_1\} \), \( p, q \in \{1, 2, \ldots, n_2\} \), \( v_p\)'s and \( v_q\)'s are not adjacent vertices in \( H_2 \), be deleted. Then, \( d(u_xv_p, u_yv_q) = \text{diam}(G) \) by a path \( u_xv_p - u_{x+1}v_p - u_{x+2}v_p - \ldots - u_iv_p - u_{i+1}v_{p+1} - \ldots - u_{y-2}v_{q-2} - u_{y-1}v_{q-1} - u_yv_q \) where \( d(u_xv_p, u_yv_q) = \text{diam}(H_1) - L \), and \( d(u_xv_p, u_yv_q) = L \).

Also, \( d(u_iv_w, u_iv_z) = \text{diam}(H_2) \) or \( d(u_iv_w, u_qv_z) = \text{diam}(H_2) + 1 \) when the distance between \( v_w, v_z \) is even or odd respectively.

Thus the \( \text{diam}(G) \) remains the same.

Hence, \( D^0(G) \geq n_1m_2 + n_2^2m_1 - (n_1m_2 + m_1n_2 + 2m_1m_2) = \)
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\[ n_2^2 m_1 - (m_1 n_2 + 2 m_1 m_2). \]

**Theorem 2.3.6.** Let \( G \cong H_1 \circ H_2 \). Then \( D^1(G) = 1 \) if and only if \( G \) is any one of the following graphs where,

(a) both \( H_1 \) and \( H_2 \) are complete graphs.

(b) \( H_1 = K_2 \) or a connected graph with diameter two in which there exist at least one pair of adjacent vertices with no path of length two between them and \( H_2 \) is a disconnected graph in which there exist at least one component with an isolated vertex.

**Proof.** (a) Let \( G \cong K_{n_1} \circ K_{n_2} \) where \( n_1, n_2 \geq 2 \). Then the deletion of any edge increases the \( \text{diam}(G) \).

(b) Suppose that \( H_1 = K_2 \) and \( H_2 \) is a disconnected graph with an isolated vertex \( v_p \), then \( \text{diam}(G) = 2 \). Let an edge \( u_i v_p - u_j v_p \), be deleted. There is a path \( u_i v_p - u_j v_q - u_i v_q - u_j v_p \) of length three between them.

Let \( H_1 \) be a connected graph with diameter two in which the adjacent vertices \( u_r, u_s \) have no path of length two between them and \( H_2 \) be a disconnected graph with an isolated vertex \( v_p \), then \( \text{diam}(G) = 2 \). Let an edge \( u_r v_p - u_s v_p \), be deleted. There is a path \( u_r v_p - u_s v_q - u_r v_q - u_s v_p \) of length three between them.
Conversely suppose that \( D^1(G) = 1 \).

Let \( u_x, u_y \) be a pair of diametral vertices in \( H_1 \), by a path
\( u_x - u_{x+1} - u_{x+2} - \ldots - u_{y-1} - u_y \) and \( v_w, v_z \) be a pair of diametral vertices in \( H_2 \), by a path
\( v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z \).

Suppose that \( H_1 \) is a complete graph and \( H_2 \) is any connected graph, then \( \text{diam}(G) \leq 2 \).

Let an edge \( u_iv_p - u_iv_q \) or \( u_iv_p - u_jv_p \) or \( u_iv_p - u_jv_q \), be deleted. There exist at least two paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus to increase the \( \text{diam}(G) \) by one, \( H_2 \) should be a complete graph. This proves (a).

Suppose that \( H_1 \) is a connected graph.

Let an edge \( u_iv_w - u_jv_w \), be deleted. If \( H_2 \) is any connected graph, then there exist at least \( \kappa(H_2) + 1 \) paths \( u_xv_w - u_{x+1}v_z \ldots u_{y-1}v_z - u_yv_w \) of length \( \text{diam}(H_1) \) between \( u_xv_w \) and \( u_yv_w \) in \( G \) where \( z \in \{1, 2, \ldots, n_2\} \). Thus, when \( H_2 \) is a connected graph, at least two edges should be deleted to increase the \( \text{diam}(G) \). Hence, it is clear that \( H_2 \) should be a disconnected
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Now, if $H_2$ is a disconnected graph without an isolated vertex, then there exist at least two paths of length $\text{diam}(G)$ between a pair of diametral vertices $u_xv_w$ and $u_yv_w$ in $G$. Thus, at least two edges should be deleted to increase the $\text{diam}(G)$. Hence, $H_2$ is a disconnected graph in which there exist at least one component with an isolated vertex.

If $\text{diam}(H_1) \geq 3$, then the deletion of an edge will not increase the $\text{diam}(G)$. There is a path of length at most three between each pair of vertices. Hence, $H_1$ is any connected graph with $\text{diam}(H_1) \leq 2$.

Let $H_1$ be a complete graph with $n_1 > 2$.

Since $n_1 > 2$ there exist at least two paths of length two between each pair of vertices in $G$. Thus, the deletion of an edge from $G$ does not increase the $\text{diam}(G)$. Hence, $n_1 = 2$.

Let $\text{diam}(H_1) = 2$.

Let an edge $u_iv_p - u_jv_p$, be deleted. Then the $\text{diam}(G)$ increases only if $u_i$ and $u_j$ have no path of length two between
them in $H_1$. Otherwise, at least two edges should be deleted to increase the diam($G$). Also, the distance between any two other vertices is not affected by the removal of these edges. Hence, $H_1$ should be a connected graph with diameter two in which there exist at least one pair of adjacent vertices with no path of length two between them.

This proves (b). \hfill \square

**Corollary 2.3.7.** $G \cong H_1 \circ H_2$ is diameter minimal if and only if $G$ is any one of the following graphs where,

(a) both $H_1$ and $H_2$ are complete graphs.

(b) $H_1 = K_2$ or a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in $H_1$ and $H_2$ is a totally disconnected graph.

**Proof.** (a) Let $G = K_{n_1} \circ K_{n_2}$. Then $G$ is diameter minimal.

(b) Suppose that $H_1$ is a $K_2$ and $H_2$ is a totally disconnected graph, then diam($G$) = 2. Let an edge $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_q$, be deleted. Then there is a path $u_iv_p - u_jv_q - u_iu_q - u_jv_p$ or $u_iv_p - u_jv_p - u_iu_q - u_jv_q$ of length three between each pair of vertices. Thus, the deletion of any edge increases the diam($G$).
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Suppose that $H_1$ is a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in $H_1$ and $H_2$ is a totally disconnected graph, then $\text{diam}(G) = 2$. Let an edge $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_q$, be deleted. There is a path of length three between these pairs of vertices. Thus, the deletion of any edge increases the $\text{diam}(G)$.

Hence, $G$ is diameter minimal.

Conversely suppose that $G$ is diameter minimal. In Theorem 2.3.6 we have characterized the lexicographic product of graphs whose diameter increases by the deletion of a single edge. Hence, we need to prove the theorem only for such $G$s.

Let $G \cong K_{n_1} \circ K_{n_2}$. Then, clearly $G$ is diameter minimal.

Suppose that $H_1 = K_2$ and $H_2$ is a disconnected graph in which there exist at least one component with an isolated vertex.

Let an edge $u_iv_p - u_iv_q$ where $v_p, v_q$ are not isolated vertices in $H_2$, be deleted. Since $v_p, v_q$ are not isolated vertices there is a path of length two between $u_iv_p$ and $u_iv_q$ in $G$. Hence, if $H_2$ contains any pair of adjacent vertices, the deletion of that edge will not increase the $\text{diam}(G)$. Thus, $H_2$ is a totally disconnected
Suppose that $H_1$ is a connected graph with diameter two in which at least one pair of adjacent vertices have no path of length two between them and $H_2$ is a disconnected graph in which there exist at least one component with an isolated vertex.

As in the previous case, if $H_2$ contains any pair of adjacent vertices, the deletion of that edge will not increase the $\text{diam}(G)$. Hence, $H_2$ is a totally disconnected graph.

Let an edge $u_i v_p - u_j v_p$ where the adjacent vertices $u_i$ and $u_j$ have a path of length two in $H_1$, be deleted. If any two adjacent vertices in $H_1$ have a path of length two between them, then the deletion of an edge will not increase the $\text{diam}(G)$. Thus, $H_1$ is a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in $H_1$.

**Theorem 2.3.8.** Let $G \cong H_1 \circ H_2$.

Then $D^1(G) \leq \alpha n_2$ where $\alpha$ is the minimum number of edge disjoint paths of length $\text{diam}(H_1)$ between any two vertices in $H_1$.

**Proof.** Follows from Theorem 2.2.6.
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**Theorem 2.3.9.** Let \( G \cong H_1 \circ H_2 \). Then \( D^{-1}(G) = 1 \) if and only if \( G \) is any one of the following graphs where,

(a) \( H_2 \) has a universal vertex and \( H_1 \) is a connected graph with \( \text{diam}(H_1) \geq 4 \) and \( D^{-2}(H_1) = 1 \) when an edge is added between a diametral vertex and any other vertex of \( H_1 \).

(b) \( H_2 \) is any graph and \( H_1 \) is a connected graph with \( \text{diam}(H_1) \geq 4 \) and \( D^{-1}(H_1) = 1 \) when an edge is added between the diametral vertices or between any two other vertices of \( H_1 \).

*Proof.* Follows from Theorem 2.2.7. \( \square \)

**Corollary 2.3.10.** There does not exist a graph \( G \cong H_1 \circ H_2 \) such that \( G \) is diameter maximal.