Chapter 5

Domination criticality in the Cartesian product of graphs

A connected dominating set is an important notion and has many applications in routing and management of networks. In this chapter we study the Cartesian product of graphs $G$ with connected domination number, $\gamma_c(G) = 2, 3$ and characterize such graphs. Also, we characterize the $k - \gamma$ - vertex (edge) critical graphs and $k - \gamma_c$ - vertex (edge) critical graphs for $k = 2, 3$ where $\gamma$ denotes the domination number of $G$. We also

\[ \text{Some results of this chapter are included in the following paper.} \]
Chapter 5. Domination criticality in the Cartesian product of graphs
discuss the vertex criticality in grids.

5.1 Domination critical graphs

Theorem 5.1.1. Let $G \cong H_1 \square H_2$ be a connected graph. Then $\gamma(G) = 2$ if and only if $H_1 = K_2$ and $H_2$ is either a $C_4$ or has a universal vertex.

Consider $G \cong K_2 \square C_4$, then a minimum dominating set of $G$ is

$D = \{u_1v_1, u_2v_3\}$. If $G \cong K_2 \square H_2$, where $H_2$ has a universal vertex $v_i$, then a minimum dominating set of $G$ is $D = \{u_1v_i, u_2v_i\}$.

Hence, $\gamma(G) = 2$ in both the cases.

Conversely suppose that $\gamma(G) = 2$.

Suppose that both $H_1$ and $H_2$ are not complete graphs. Then, $\gamma(G) \geq min\{|H_1|, |H_2|\} \geq 3$.

Hence, at least one graph (say) $H_1$ should be complete.

Let $G \cong K_{n_1} \square H_2$.

Suppose that $H_1$ is a complete graph of order at least three. If $H_2$ has a universal vertex, then a minimum dominating set of $G$ con-
5.1. Domination critical graphs

contains vertices from each layer of $G$ and $3 \leq \gamma(G) \leq \min\{n_1, n_2\}$. If $H_2$ does not have a universal vertex, then $\gamma(H_2) \geq 2$ and a minimum dominating set of $G$ contains vertices from each layer of $G$ and $3 \leq \gamma(G) \leq n_1$. Thus, in both the cases $\gamma(G) \geq 3$. Hence, $n_1 = 2$. Thus, $G \cong K_2 \Box H_2$.

Let $n_2 \geq 2$.

Then $\gamma(G) \leq \min\{2\gamma(H_2), n_2\gamma(K_2)\} = \min\{2\gamma(H_2), n_2\}$ (1).

From (1) we have $\gamma(G) = 2$ only when $H_2$ has a universal vertex, since $n_2 \geq 2$.

Next, we consider the case when $\gamma(H_2) \geq 2$.

Let $n_2 \geq 5$.

Suppose that $H_2$ contains a vertex $v_i$ of degree $(n_2 - 2)$ and $v_i$ is not adjacent to $v_j$, then $\gamma(H_2) = 2$. Now, a minimum dominating set of $G$ is $D = \{u_1v_i, u_2v_i, u_1v_j\}$ and $\gamma(G) = 3$.

Suppose that $H_2$ contains a vertex of degree at most $(n_2 - 3)$, then $\gamma(H_2) = 2$. Let $v_p$ be a vertex of degree $(n_2 - 3)$ and is not adjacent to $v_q$ and $v_r$ in $H_2$. Then, in $G$ the vertices $u_1v_i$ and $u_2v_i$ dominate $2n_2 - 4$ vertices and the remaining four vertices $u_1v_q, u_1v_r, u_2v_q$ and $u_2v_r$ cannot be dominated by a single vertex. Hence, in these cases $\gamma(G) \geq 3$. Thus, $n_2 \leq 4$. 
Chapter 5. Domination criticality in the Cartesian product of graphs

Now, by an exhaustive verification of all graphs with \( n_2 \leq 4 \) it follows that \( G \cong K_2 \square C_4 \).

**Illustration of Theorem 5.1.1**

![Illustration](image)

Fig 5.1: (i) \( G = K_2 \square C_4 \), \( \gamma(G) = 2 \) (ii) \( G = K_2 \square K_{1,4} \), \( \gamma(G) = 2 \).

**Corollary 5.1.2.** Let \( G \cong H_1 \square H_2 \) be a connected graph. Then \( G \) is 2 - \( \gamma \) - vertex critical if and only if \( G = C_4 \).

**Proof.** In Theorem 5.1.1 we have characterized the Cartesian product of graphs with \( \gamma(G) = 2 \). Hence, we need to prove the theorem only for such \( G \)s.

First, note that \( G \cong K_2 \square C_4 \) is not 2 - \( \gamma \) - vertex critical.

Now, consider \( G \cong K_2 \square K_{n_2} \), where \( n_2 \geq 3 \). Then, a minimum dominating set \( D = \{u_1v_x, u_2v_x\} \) of \( G \) contains a vertex from each layer of \( K_{n_2} \). Now, let a vertex \( u_iv_p \) where \( p \in \{1, 2, \ldots, n_2\} \), be deleted. If \( p = x \), then we can find another
minimum dominating set $D = \{u_1 v_p, u_2 v_y\}$. If $p \neq x$, then the minimum dominating set $D = \{u_1 v_x, u_2 v_x\}$ of $G$ remains the same. Thus, in both the cases $\gamma(G - v) = \gamma(G) = 2 \forall v \in V(G)$. Hence, $H_2 = K_2$.

Consider $G \cong K_2 \square H_2$ where $H_2$ is a not complete graph with a universal vertex $v_p$. Then, a minimum dominating set $D = \{u_1 v_p, u_2 v_p\}$ of $G$ contains a vertex from each layer of $H_2$. Now, let a vertex $u_1 v_q$ where $q \in \{1, 2, ..., n_2\}$, be deleted. If $p \neq q$, then the minimum dominating set $D = \{u_1 v_x, u_2 v_x\}$ of $G$ remains the same. If $p = q$, then in $G$ the vertex $u_2 v_q$ dominate the $n_2$ vertices $u_2 v_i$ and the remaining $n_2$ vertices cannot be dominated by a single vertex, since we have deleted the universal vertex from the layer of $H_2$. Hence, $\gamma(G) \geq 3$. Thus, $G \cong K_2 \square H_2$ is not $2$ - $\gamma$ - vertex critical. \qed

**Corollary 5.1.3.** Let $G \cong H_1 \square H_2$ be a connected graph. Then $G$ is $2$ - $\gamma$ - edge critical if and only if $G = C_4$.

**Proof.** In Theorem 5.1.1 we have characterized the Cartesian product of graphs with $\gamma(G) = 2$. Hence, we need to prove the theorem only for such $G$s.
Chapter 5. Domination criticality in the Cartesian product of graphs

First, note that $G \cong K_2 \Box C_4$ is not 2 - $\gamma$ - edge critical.

Consider $G \cong K_2 \Box H_2$, where $H_2$ is a not complete graph with a universal vertex or a complete graph with $n_2 \geq 3$. Let an edge $u_1v_p - u_2v_i$ where $i \in \{1, 2, 3, \ldots, n_2\}$, be added. Then, the addition of an edge does not make either $G$ a complete graph or a graph with a universal vertex. Thus, $\gamma(G)$ remains the same. Hence, $H_2 = K_2$. \hfill \Box

**Corollary 5.1.4.** Let $G \cong H_1 \Box H_2$ be a connected graph. Then $\gamma_e(G) = \gamma(G) = 2$ if and only if $H_1 = K_2$ and $H_2$ has a universal vertex.

*Proof.* It suffices to show that the dominating set of $G$ in Theorem 5.1.1 is connected.

Consider $G \cong K_2 \Box C_4$. Then a minimum dominating set of $G$ is $D = \{u_1v_1, u_2v_3\}$ and $\gamma(G) = 2$. From Fig 5.1, it is clear that, $\langle D \rangle$ is disconnected.

Consider $G \cong K_2 \Box H_2$ where $H_2$ is a complete graph or a not complete graph with a universal vertex $v_p$. Then a minimum dominating set of $G$ is $D = \{u_1v_p, u_2v_p\}$ and $\langle D \rangle$ is connected. Hence, $\gamma_e(G) = \gamma(G) = 2$. \hfill \Box
Corollary 5.1.5. Let $G \cong H_1 \square H_2$ be a connected graph. Then $G$ is $2 - \gamma_c$-vertex critical if and only if $G = C_4$.

Corollary 5.1.6. Let $G \cong H_1 \square H_2$ be a connected graph. Then $G$ is $2 - \gamma_c$-edge critical if and only if $G = C_4$.

Theorem 5.1.7. Let $G \cong H_1 \square H_2$ be a connected graph. Then $\gamma(G) = 3$ if and only if $G$ is the Cartesian product of any one of the following graphs where,

(a) $H_1 = K_3$ or $P_3$ and $H_2$ has a universal vertex.
(b) $H_1 = K_2$ and $H_2$ has a vertex of degree $n_2 - 2$.
(c) $H_1 = K_2$ and $H_2$ has a vertex $v_r$ of degree $n_2 - 3$ and is not adjacent to the vertices $v_p$ and $v_q$ with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$.
(d) $H_1 = K_3$ or $P_3$ and $H_2 = C_4$.

Proof. Let $G \cong H_1 \square H_2$ where $H_1$ is a $K_3$ or $P_3$ and $H_2$ has a universal vertex $v_i$. Then, a minimum dominating set of $G$ is $D = \{u_1v_i, u_2v_i, u_3v_i\}$ and $\gamma(G) = 3$.

If $G \cong K_2 \square H_2$ where $H_2$ has a vertex $v_j$ of degree $n_2 - 2$ and $v_j$ is not adjacent to $v_p$ in $H_2$, then a minimum dominating set of $G$ is $D = \{u_1v_j, u_2v_j, u_1v_p\}$ and $\gamma(G) = 3$.

Further if $G \cong K_2 \square H_2$ where $H_2$ has a vertex $v_r$ of degree
Chapter 5. Domination criticality in the Cartesian product of graphs

$n_2 - 3$ and is not adjacent to the vertices $v_p$ and $v_q$ with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$, then a minimum dominating set of $G$ is $D = \{u_1v_r, u_2v_p, u_2v_q\}$ and $\gamma(G) = 3$.

Now, $G \cong H_1 \square C_4$ where $H_1$ is a $K_3$ or $P_3$, then a minimum dominating set of $G$ is $D = \{u_1v_1, u_2v_3, u_3v_1\}$ and $\gamma(G) = 3$.

Conversely suppose that $\gamma(G) = 3$.

(I) Suppose that both $H_1$ and $H_2$ are complete graphs, where $n_1, n_2 \geq 4$.

Then $\gamma(G) \geq min\{4, 4\} = 4$. Thus, at least one graph (say) $H_1$ has order $n_1 \leq 3$. But $\gamma(G) = 3$ only when $H_1$ is a $K_3$. Hence, $G \cong K_3 \square K_{n_2}$ where $n_2 \geq 3$.

(II) Suppose that $H_1$ is a complete graph and $H_2$ is not a complete graph.

If $n_1, n_2 \geq 4$, then $\gamma(G) \geq 4$. Thus, to prove the theorem we have to consider the following cases.

(1) Let $n_1 = 2$ and $n_2 = 3$, then $\gamma(G) = 2$.

(2) Let $n_1 = 2$ and $n_2 \geq 4$. 

Consider $G \cong K_2 \sqcap H_2$. From (1) we have $\gamma(G) \leq \min\{2\gamma(H_2), n_2\}$. Thus it is clear that we do not have to consider the case when $\gamma(H_2) = 1$, since $\gamma(G) = 3$. Hence, $\gamma(H_2) \geq 2$.

If $\gamma(H_2) \geq 3$, then $\gamma(G) \geq 4$. Hence, we need consider only the case when $\gamma(H_2) = 2$.

Now, suppose that $H_2$ is not a complete graph with $\gamma(H_2) = 2$.

Suppose that a minimum dominating set of $H_2$ is $D = \{v_p, v_q\}$.

Let a minimum dominating set of $G$ be $D = \{u_1v_p, u_1v_q, u_2v_p\}$. The vertices $u_1v_p$ and $u_1v_q$ dominate $n_2 + 2$ vertices in $G$. Now, the remaining $2n_2 - (n_2 + 2) = n_2 - 2$ vertices will be dominated by a single vertex $u_2v_p$, only if $\deg(v_p) = n_2 - 2$. Hence, $H_2$ has a vertex of degree $n_2 - 2$. This, proves (b).

Let a minimum dominating set of $G$ contain a vertex $u_1v_r$, where $v_r$ is not a neighbour of $v_p$ and $v_q$ in $H_2$. The vertex $u_1v_r$ dominate the $n_2 - 1$ vertices $u_1v_x$ and $u_2v_r$, where $x \neq r \in \{1, 2, ..., n_2\}$ in $G$. If the dominating set contain the vertex $u_2v_r$, then the vertices $u_1v_r$ and $u_2v_r$ dominate $2n_2 - 4$ vertices in $G$. The remaining four vertices $u_1v_q$, $u_1v_q$, $u_2v_p$, and $u_2v_q$ cannot be dominated by a single vertex and hence $\gamma(G) \geq 3$. Thus,
Chapter 5. Domination criticality in the Cartesian product of graphs

The dominating set does not contain the vertex $u_2v_r$. Since, a minimum dominating set of $G$ contain the vertex $u_1v_r$ and $v_r$ is not a neighbour of $v_p$ and $v_q$ in $H_2$, the dominating set of $G$ should contain the vertices $u_2v_p$ and $u_2v_q$. Now, the remaining $2n_2 - (n_2 - 1) = n_2 + 1$ vertices will be dominated by the vertices $u_2v_p$ and $u_2v_q$, only if $N[v_p] \cup N[v_q] = V(H_2) - v_r$. Hence, $H_2$ has a vertex $v_r$ of degree $n_2 - 3$ and is not adjacent to the vertices $v_p$ and $v_q$ with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$. This, proves (c).

(3) By an exhaustive verification of all graphs with $n_2 = 4$ it follows that $G \cong K_3 \square C_4$.

(4) Let $n_1 = 3$ and $n_2 \geq 4$.

Consider $G \cong K_3 \square H_2$. Let $\gamma(H_2) \geq 2$, then $\gamma(G) \geq 4$. Thus, $H_2$ has a universal vertex.

(5) Let $n_2 = 3$ and $n_1 \geq 3$.

Consider $G \cong K_{n_1} \square P_3$, then a minimum dominating set of $G$ is $D = \{u_1v_1, u_1v_2, u_1v_3\}$. The vertex $u_1v_1$ dominate the vertices $u_iv_1$ where $i \in \{1, 2, \ldots, n_1\}$, since $H_1$ is a complete graph. Similarly, the vertices $u_1v_2$ and $u_1v_3$ dominate the remaining
vertices in $G$. Thus, $\gamma(G) = 3$.

(III) Suppose that both $H_1$ and $H_2$ are not complete graphs. If $n_1, n_2 \geq 4$, then $\gamma(G) \geq 4$. Hence, $n_1 = 3$ and $n_2 \geq 4$. If $\gamma(H_2) \geq 3$, clearly $\gamma(G) \geq 4$. Hence, the domination number of $H_2$ is at most 2.

We know that $\gamma(G) \leq \min\{3\gamma(H_2), n_2\}$.

If $\gamma(H_2) = 1$ where $v_i$ is a universal vertex in $H_2$, then $\gamma(G) = 3$. Hence, $G \cong P_3 \Box H_2$ where $H_2$ has a universal vertex.

Now, suppose that $\gamma(H_2) = 2$ and $n_2 \geq 6$, then by a similar argument of II(2) it follows that $\gamma(G) \geq 4$. Hence, $n_2 \leq 5$.

By an exhaustive verification of all graphs with $n_2 = 3, 4, 5$ it follows that $G \cong P_3 \Box C_4$. □

**Illustration of Theorem 5.1.7.**

Fig 5.2: (a) $G = P_3 \Box K_{1,3}$, $\gamma(G) = 3$.
(b) $G = K_2 \Box C_4$, $\gamma(G) = 3$.
(c) $G = K_2 \Box K_{3,3}$, $\gamma(G) = 3$.
(d) $G = P_3 \Box C_4$, $\gamma(G) = 3$. 
Chapter 5. Domination criticality in the Cartesian product of graphs

Corollary 5.1.8. Let $G \cong H_1 \square H_2$ be a connected graph. Then $G$ is a $3$ - $\gamma$ - vertex critical graph if and only if $H_1 = H_2 = K_3$.

Proof. It suffices to prove that $\gamma(G - v) < \gamma(G) \forall v \in G$ of Theorem 5.1.7.

Consider $G \cong K_3 \square K_{n_2}$ where $n_2 \geq 4$. Then, a minimum dominating set $D = \{u_1v_x, u_2v_x, u_3v_x\}$ of $G$ contains a vertex from each layer of $K_{n_2}$. Now, let a vertex $u_i v_p$ where $p \in \{1, 2, ..., n_2\}$, be deleted. If $p = x$, then we can find another minimum dominating set $D = \{u_1v_y, u_2v_y, u_3v_y\}$. If $p \neq x$, then the minimum dominating set $D = \{u_1v_z, u_2v_z, u_3v_z\}$ of $G$ remains the same. Thus, in both cases $\gamma(G - v) = \gamma(G) = 3 \forall v \in V(G)$. Hence, $H_2 = K_3$.

Consider $G \cong K_3 \square H_2$ or $G \cong P_3 \square H_2$ where $H_2$ has a universal vertex $v_i$. Then, a minimum dominating set $D = \{u_1v_i, u_2v_i, u_3v_i\}$ of $G$ contains a vertex from each layer of $H_2$. Now, let a vertex $u_1v_q$ where $q \in \{1, 2, ..., n_2\}$, be deleted. If $i \neq q$ then, the minimum dominating set $D = \{u_1v_i, u_2v_i, u_3v_i\}$ of $G$ remains the same. If $p = i$, then in $G$, the vertices $u_2v_i$ and $u_3v_i$ dominate the $2n_2$ vertices and the remaining $n_2$ vertices
$u_1v_x$, where $q \in \{1, 2, ..., n_2\}$ cannot be dominated by a single vertex, since we have deleted the universal vertex from the layer of $H_2$. Hence, $\gamma(G) \geq 3$. Thus $\gamma(G - v) \geq \gamma(G) \forall v \in V(G)$. Hence, $G \cong K_3 \boxtimes H_2$ or $G \cong P_3 \boxtimes H_2$ is not $3$ - $\gamma$ - vertex critical.

Consider $G \cong K_2 \boxtimes H_2$ where $H_2$ has a vertex $v_i$ of degree $(n_2 - 2)$ and is not adjacent to the vertex $v_j$. Then, a minimum dominating set of $G$ is $D = \{u_1v_i, u_2v_i, u_1v_j\}$ and $\gamma(G) = 3$. Now, let a vertex $u_1v_q$ where $q \in \{1, 2, ..., n_2\}$, be deleted. If $i \neq q$, then the minimum dominating set $D = \{u_1v_i, u_2v_i, u_1v_j\}$ of $G$ remains the same. If $q = i$, then in $G$, the vertices $u_2v_i$ and $u_1v_j$ dominate the $n_2 + 1$ vertices and the remaining $n_2 - 1$ vertices $u_1v_x$ cannot be dominated by a single vertex, since we have deleted the vertex $u_1v_i$ from the layer of $H_2$. Hence, $\gamma(G) \geq 3$ and $G \cong K_2 \boxtimes H_2$, where $H_2$ has a vertex $v_i$ of degree $(n_2 - 2)$, is not $3$ - $\gamma$ - vertex critical.

Consider $G \cong K_2 \boxtimes H_2$, where $H_2$ has a vertex $v_p$ of degree at most $(n_2 - 3)$ and $v_p$ is not adjacent to $v_q$ and $v_r$. Then then by a similar argument, as in the above case, it follows that $\gamma(G) \geq 3$. Hence, $G \cong K_2 \boxtimes H_2$, where $H_2$ has a vertex of degree at most $(n_2 - 3)$, is not $3$ - $\gamma$ - vertex critical.
Chapter 5. Domination criticality in the Cartesian product of graphs

Consider $G \cong K_3 \Box C_4$, $G \cong P_3 \Box C_4$ and $G \cong K_2 \Box C_5$. In all these cases $G$ is not $3 \cdot \gamma$ - vertex critical. \hspace{1cm} \Box

**Corollary 5.1.9.** Let $G \cong H_1 \Box H_2$ be a connected graph. Then $G$ is a $3 \cdot \gamma$ - edge critical graph if and only if $H_1 = H_2 = K_3$.

**Proof.** It suffices to prove that $\gamma(G + e) = 2 \forall e \notin G$ of Theorem 5.1.7.

Consider $G \cong K_3 \Box H_2$ or $G \cong P_3 \Box H_2$, where $H_2$ has a universal vertex $v_1$ and $n_2 \geq 4$. Then, a minimum dominating set of $G$ is $D = \{u_1v_1, u_2v_1, u_3v_1\}$. In $G$, the vertex $u_1v_1$ dominate the $n_2$ vertices $u_1v_i$, where $i \in \{1, 2, 3, \ldots , n_2\}$ and $u_2v_1$ dominate the $n_2$ vertices $u_2v_i$, where $i \in \{1, 2, 3, \ldots , n_2\}$. Let an edge $u_1v_1 - u_2v_p$, be added. Then, in $G$ the vertex $u_1v_1$ dominate the $n_2 + 1$ vertices $u_1v_i$, $u_2v_p$, where $i \in \{1, 2, 3, \ldots , n_2\}$ and $u_2v_1$ dominate the $n_2 - 1$ vertices $u_2v_i$, where $i \neq p \in \{1, 2, 3, \ldots , n_2\}$ and $u_3v_1$ dominate the $n_2$ vertices $u_3v_i$, where $i \in \{1, 2, 3, \ldots , n_2\}$. Hence, the minimum dominating set $D = \{u_1v_1, u_2v_1, u_3v_1\}$ of $G$ remains the same. Thus, $n_2 = 3$. By an exhaustive verification of all such graphs, it follows $G$ is a $3 \cdot \gamma$ - edge critical graph if and only if $H_1 = H_2 = K_3$. 
Consider $G \cong K_2 \Box H_2$, where $H_2$ has a vertex $v_i$ of degree $n_2 - 2$ and $v_i$ is not adjacent to $v_j$. Then, a minimum dominating set of $G$ is $D = \{u_1v_i, u_2v_i, u_1v_j\}$. In $G$, the vertex $u_1v_i$ dominate the $n_2 - 1$ vertices $u_1v_p$, where $p \neq j \in \{1, 2, 3, \ldots, n_2\}$ and $u_2v_i$ dominate the $n_2 - 1$ vertices $u_2v_p$, where $p \neq j \in \{1, 2, 3, \ldots, n_2\}$. Let an edge $u_1v_i - u_2v_j$, be added. Then, in $G$ the vertex $u_1v_i$ dominate the $n_2$ vertices $u_1v_p$, $u_2v_j$, where $p \neq j \in \{1, 2, 3, \ldots, n_2\}$ and $u_2v_1$ dominate the $n_2 - 1$ vertices $u_2v_p$, where $p \neq j \in \{1, 2, 3, \ldots, n_2\}$ and the remaining one vertex $u_1v_j$ is not dominated by the vertices $u_1v_i$ and $u_2v_i$. Hence, the minimum dominating set $D = \{u_1v_i, u_2v_i, u_1v_j\}$ of $G$ remains the same. Thus, $G \cong K_2 \Box H_2$, where $H_2$ has a vertex $v_i$ of degree $n_2 - 2$, is not $3$ - $\gamma$ - edge critical.

Consider $G \cong K_2 \Box H_2$, where $H_2$ has a vertex $v_p$ of degree at most $(n_2 - 3)$ and $v_p$ is not adjacent to $v_q$ and $v_r$. Then, by a similar argument, as in the above case, it follows that $\gamma(G) = 3$. Hence, $G \cong K_2 \Box H_2$ where $H_2$ has a vertex of degree at most $(n_2 - 3)$, is not $3$ - $\gamma$ - edge critical.

In all other cases, $G$ is not $3$ - $\gamma$ - edge critical.

**Corollary 5.1.10.** Let $G \cong H_1 \Box H_2$ be a connected graph. Then
\[ \gamma_c(G) = \gamma(G) = 3 \text{ if and only if } H_1 = K_3 \text{ or } P_3 \text{ and } H_2 \text{ has a universal vertex.} \]

**Proof.** It suffices to prove that the dominating set of \( G \) in Theorem 5.1.7 is connected.

Consider \( G \cong K_3 \Box H_2 \) or \( G \cong P_3 \Box H_2 \), where \( H_2 \) has a universal vertex \( v_i \).

Then, a minimum dominating set of \( G \) is \( D = \{u_1v_i, u_2v_i, u_3v_i\} \) and \( \gamma(G) = 3 \). Also, \( < D > \) is connected. Hence, \( \gamma_c(G) = 3 \).

Consider \( G \cong K_2 \Box H_2 \), where \( H_2 \) has a vertex \( v_j \) of degree \( (n_2 - 2) \) and \( v_j \) is not adjacent to \( v_x \).

Then, a minimum dominating set of \( G \) is \( D = \{u_1v_j, u_2v_j, u_1v_x\} \) and \( \gamma(G) = 3 \). Also, \( < D > \) is disconnected, since \( v_j \) is not adjacent to \( v_x \) in \( H_2 \). Hence, \( \gamma_c(G) \geq 4 \).

Consider \( G \cong K_2 \Box H_2 \), where \( H_2 \) has a vertex \( v_p \) of degree \( (n_2 - 3) \) and \( v_p \) is not adjacent to \( v_q \) and \( v_r \) with \( N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2) \).

Then, a minimum dominating set of \( G \) is \( D = \{u_1v_p, u_2v_q, u_2v_r\} \) and \( \gamma(G) = 3 \). Also, \( < D > \) is disconnected, since \( v_p \) is not adjacent to \( v_q \) and \( v_r \) in \( H_2 \). Hence, \( \gamma_c(G) \geq 4 \).
5.2. Vertex criticality in grids

In all other cases, \( \gamma(G) = 3 \) and \( <D> \) is disconnected. Hence, \( \gamma_c(G) \geq 4 \).

**Corollary 5.1.11.** Let \( G \cong H_1 \square H_2 \) be a connected graph. Then \( G \) is a \( 3 - \gamma_c \) - vertex critical graph if and only if \( H_1 = H_2 = K_3 \).

**Corollary 5.1.12.** Let \( G \cong H_1 \square H_2 \) be a connected graph. Then \( G \) is a \( 3 - \gamma_c \) - edge critical graph if and only if \( H_1 = H_2 = K_3 \).

5.2 Vertex criticality in grids

**Theorem 5.2.1.** Let \( G \cong P_{n_1} \square P_{n_2} \). Then \( G \) is vertex critical if and only if \( G \cong P_2 \square P_2 \).

**Proof.** It suffices to prove the converse.

Let \( G \cong P_{n_1} \square P_{n_2} \), where \( n_1, n_2 \geq 3 \).

Let \( u_i v_j \in D \), where \( i \neq 1, n_1 \). Since, each vertex in \( D \) will dominate at most five vertices, it will dominate two vertices from the \( P_{n_2} \) - layer at \( u_i \) and two vertices each from the \( P_{n_2} \) - layer at \( u_{i-1} \) and \( P_{n_2} \) - layer at \( u_{i+1} \), where \( u_{i-1}, u_{i+1} \) are the neighbours of \( u_i \) in \( P_{n_1} \). Let a vertex \( u_i v_{j-1} \), be deleted.
Then, the minimum dominating set \( D \) of \( G \) remains the same. Hence, \( G \) is not a vertex critical graph. Thus, \( n_1 = n_2 = 2 \).

**Theorem 5.2.2.** Let \( G \cong P_{n_1} \Box P_{n_2} \). If \( n_1, n_2 \geq 4 \), then a minimal dominating set of \( G \) is disconnected.

**Proof.** Let \( u_i v_j \in D \). Since, the maximum degree of a vertex in \( G \) is four, each vertex in \( D \) will dominate at most five vertices, it will dominate two vertices from the \( P_{n_2} \)-layer at \( u_i \) and two vertices each from the \( P_{n_2} \)-layer at \( u_{i-1} \) and \( P_{n_2} \)-layer at \( u_{i+1} \), where \( u_{i-1}, u_{i+1} \) are the neighbours of \( u_i \) in \( P_{n_1} \).

Now, suppose that the vertex \( u_p v_{j+1} \in D \). If \( p \neq i \), then \( < D > \) is disconnected. If \( p = i \), then \( D \) should contain a vertex either from \( u_{i-1} v_x - u_{i-1} v_y \) or from \( u_{i+1} v_x - u_{i+1} v_y \), since \( n_1, n_2 \geq 4 \). Then, \( < D > \) is disconnected.

**Corollary 5.2.3.** Let \( G \cong P_{n_1} \Box P_{n_2} \). Then \( \gamma_c(G) = \gamma(G) \) if and only if \( G \) is any one of the following graphs where,

(a) \( G \cong P_3 \Box P_2 \).
(b) \( G \cong P_3 \Box P_3 \).
(c) \( G \cong P_3 \Box P_3 \).
(d) \( G \cong P_3 \Box P_4 \).