Chapter 5

Single Server Queue with Several Services

In the previous chapters we considered interruption of service either by self-interruption (chapter 2), feedback (chapter 3) or through arrival of higher priority customers (chapter 4). In present and the chapter to follow, we analyze cases where permanent interruption (removal from service) takes place due to erroneous service offered or exactly needed service is offered after going through one or more undesired service. That is to say the previous chapters we followed the conventional assumption that the server is completely aware of the exact service requirement of a customer and customer is sure about the type of service he needs. The present and the next chapter discuss models where service requirement of a customer is exactly not known to him nor to the server(s) since a number of distinct services are offered by the service provider. For example patients approach a physician for medical help. The patient may not be aware of his exact health problem, nor the physician be able to diagnose it correctly. Quite often only one type of service is offered by the system and so conflict does not occur. In real life there are several service providing systems offering a multitude
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of service. The service may start inappropriate and will turn correct. There is also a chance of this service to continue in the incorrect mode and becomes an unsuccessful service. In the latter case the result could be disastrous, especially when life models are considered. The customer may even lose his life. We label such models as diagnostic problems and try to find out a solution to reduce the dilemma caused by this uncertainty.

Consider the example of a multi specialty hospital. A patient could be directed to a physician who has nothing to do with the patient’s ailment. However, he still starts medication - as per his diagnosis; the patient and/ physician subsequently realizes that the nature of medication the patient needed was different and refers to some other physician of a different specialty. Here again the patient may end up in the same situation as in the first case. This process could go on until either the patient or physician arrive at the exact nature of medication or the patient reaches such a condition where no medication would work from that time point on. Even in a hospital/ clinic with a single physician the above described is a probable situation.

First we analyze the above described situation in a single server set up. A service system with a preliminary service and a main service is then examined which is found to be on similar lines. This model is then identified with that of Madan [46] and Medhi [48]. We employed arbitrarily distributed service time in certain special cases of the model discussed and analyze such system using supplementary variables [19] to produce a CTMC.

Rest of the chapter is organized as follows. The mathematical model is described in section 1. This section also provides the steady-state analysis and some performance measures. Various cases of the model are considered in Sections 2 and 3. An illustration of the problem is given in Section 4. Numerical example is described in Section 5. In Section 6 we extend the analysis in the case of arbitrarily distributed service time for the undesired and desired stages of service.
5.1 The MAP/PH/1 model

The assumptions leading to the formulation of the mathematical model are

- An infinite capacity queueing system where a single server is providing different kinds of service.

- Arrival of customers to the system is according to the MAP (Markovian arrival process). In a MAP, the customers arrival is directed by an irreducible CTMC (continuous time Markov chain) \( \{\phi_t, t \geq 0\} \) with the state space \( \{1, 2, ..., m\} \). The transition intensities of the Markov chain \( \{\phi_t, t \geq 0\} \) which are accompanied by arrival of \( k(=0,1) \) customers are described by the matrices \( D_k \). Vector \( \eta \) of the stationary distribution of the process \( \{\phi_t, t \geq 0\} \) is the unique solution to the system

\[
\eta(D_0 + D_1) = \eta D = 0 \quad \text{and} \quad \eta e = 1.
\] (5.1)

Fundamental rate \( \lambda \) of the MAP is given by \( \lambda = \eta D_1 e \).

- A customer is selected for desired (required) service with probability \( p \) or to the undesired (incorrect) service with probability \( q = 1 - p \).

- PH-representation \( (\beta_1, S_1) \) of order \( n_1 \) gives the duration of the correct service time distribution when the service of a customer starts in correct service mode. Let \( S_1^0 \) be such that \( S_1 e + S_1^0 = 0 \). Let \( \mu_1^* = \beta_1 (-S_1)^{-1} e \) be the mean of this PH-representation.

- PH-representation \( (\beta_2, S_2) \) of order \( n_2 \) gives the duration of the incorrect service time distribution when the service of a customer starts in incorrect service mode. The rate (vector) of loss of customers is then given by \( S_2^0 \) and the rate (vector) of getting into correct service mode is given by \( \hat{S}_2^0 \). Note that \( S_2 e + S_2^0 + \hat{S}_2^0 = 0 \). Let \( \mu_2^* = \beta_2 (-S_2)^{-1} e \) be the mean of this PH-representation. A random threshold clock(timer) starts ticking from
the beginning of service so that the customer is pushed out of the system if the clock expires before service completion in undesired service. This timer determines the vector \( S_2^0 \).

- PH-representation \((\beta_3, S_3)\) of order \(n_3\) gives the duration of the correct service time distribution when the customer has gone through incorrect service initially. Let \( S_3^0 \) be such that \( S_3^0 e + S_3^0 = 0 \). Let \( \mu_3' = \beta_3 (-S_3)^{-1} e \) be the mean of this PH-representation.

- Under the above assumptions the service time of a customer can be modeled as a PH-distribution with representation \((\beta, S)\) of order \(n = n_1 + n_2 + n_3\), where

\[
\beta = (p\beta_1, q\beta_2, 0) \tag{5.2}
\]

\[
S = \begin{pmatrix}
S_1 & O & O \\
O & S_2 & S_3^0 \beta_3 \\
O & O & S_3
\end{pmatrix}.
\]

Let \( S^0 \) be such that \( Se + S^0 = 0 \) and \( S^0 = \begin{bmatrix} S^0_1 & S^0_2 & S^0_3 \end{bmatrix}^T \).

Let \( N(t) \) be the number of customers in the system, \( N^*(t) \) the nature of service going on—whether direct admission to required/ undesired or one that came from undesired service—designated by 1,2 and 3 respectively, \( S(t) \) the phase of service and \( A(t) \) the phase of arrival at time \( t \). With these the process \( \{(N(t), N^*(t), S(t), A(t)), t \geq 0\} \) is a continuous time Markov chain with state space \( \Omega = \{0, 1, 2, \ldots\} \), where

\[
0 = \{(0, r)/1 \leq r \leq m\}
\]

(in the level zero we need consider only the phase of arrival) and

\[
\mathbb{I} = \{(i, j, k, r)/1 \leq j \leq 3, 1 \leq k \leq n_j, 1 \leq r \leq m\}, i \geq m.
\]
Thus the infinitesimal generator of this CTMC is a LIQBD of the form

\[
Q = \begin{pmatrix}
D_0 & A_{01} \\
A_{10} & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
... & ... & ...
\end{pmatrix},
\]

(5.3)

where \( A_{01} = \beta \otimes D_1 \), \( A_{10} = S^0 \otimes I_m \), \( A_0 = I_n \otimes D_1 \), \( A_1 = S \oplus D_0 \), \( A_2 = S^0 \beta \otimes I_m \).

### 5.1.1 Stability condition

Consider \( A = A_0 + A_1 + A_2 \), the generator matrix of the Markov chain corresponding to the phase changes.

\[
A = (S + S^0 \beta) \oplus D
\]

\[
= \begin{pmatrix}
(pS_1^0 \beta_1 + S_1) \oplus D & qS_2^0 \beta_2 \otimes I_m & 0 \\
pS_2^0 \beta_1 \otimes I_m & (qS_2^0 \beta_2 + S_2) \oplus D & S_2^0 \beta_3 \otimes I_m \\
pS_3^0 \beta_1 \otimes I_m & qS_3^0 \beta_2 \otimes I_m & S_3 \oplus D
\end{pmatrix}.
\]

Let \( \pi = (\pi_1, \pi_2, \pi_3) \) be the steady-state probability vector of \( (S + S^0 \beta) \). Then

\[
\pi (S + S^0 \beta) = 0 \text{ and } \pi e = 1.
\]

(5.4)

From the relation \( \pi (S + S^0 \beta) = 0 \) we have

\[
\pi_1 (pS_1^0 \beta_1 + S_1) + \pi_2 pS_2^0 \beta_1 + \pi_3 pS_3^0 \beta_1 = 0,
\]

(5.5)

\[
\pi_1 qS_1^0 \beta_2 + \pi_2 (qS_2^0 \beta_2 + S_2) + \pi_3 qS_3^0 \beta_2 = 0,
\]

(5.6)

\[
\pi_2 S_2^0 \beta_3 + \pi_3 S_3 = 0.
\]

(5.7)

Multiplying equation (5.7) by \( e \) on right hand side we get

\[
\pi_3 S_3^0 = \pi_2 S_2^0.
\]

(5.8)
Putting this in equation (5.5) yields

\[ \pi_1 S_1^0 = -\frac{p}{q} \pi_2 S_2 e. \quad (5.9) \]

Substitute relations (5.8) and (5.9) in equation (5.6) to get

\[ \pi_2 (S_2^0 / \beta_2 + \hat{S}_2^0 / \beta_2 + S_2) = 0. \]

This implies, for an arbitrary constant \( c \),

\[ \pi_2 = c \beta_2 (-S_2)^{-1}. \quad (5.10) \]

Then from (5.9) we get

\[ \pi_1 = \frac{cp}{q} \beta_1 (-S_1)^{-1}. \quad (5.11) \]

Let \( \delta = \beta_2 (-S_2)^{-1} \hat{S}_2^0 \) be the probability that a customer, starting with incorrect service, leaves the system after getting correct service. Then the relation (5.8) gives

\[ \pi_3 = c \delta \beta_3 (-S_3)^{-1}. \quad (5.12) \]

From the normalizing condition \( \pi e = 1 \), the value of \( c \) is computed as

\[ c = \left[ \frac{p}{q} \mu'_1 + \mu'_2 + \delta \mu'_3 \right]^{-1}. \quad (5.13) \]

Now from (5.1) and (5.4) we get the steady-state probability vector of \( A \) as

\[ \hat{\pi} = \pi \otimes \eta. \]

**Theorem 5.1.1.** The stability of the system is given by

\[ \lambda < (\pi \otimes \eta)(S^0 / \beta \otimes I_m)e. \quad (5.14) \]

**Proof.** The queueing system under study with the LIQBD type generator given in (5.3) is stable if and only if rate of left drift is less than the rate of right drift, that is,

\[ \hat{\pi} A_0 e < \hat{\pi} A_2 e. \quad (5.15) \]
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The left drift rate is $\hat{\pi}(I_n \otimes D_1)e$ which when simplified reduces to $\lambda$. Now, the right drift rate is $(\pi \otimes \eta)(S^0 \beta \otimes I_m)e$.

Let $\rho = \frac{\lambda}{(\pi \otimes \eta)(S^0 \beta \otimes I_m)e}$. Then from (5.14), we have $\rho < 1$. \hfill $\square$

5.1.2 Steady-state probability vector

A brief outline for the computation of the stationary probability vector of the system is as follows. Let $x$ denote the steady-state probability vector of the generator $Q$. Then

$$xQ = 0 \text{ and } xe = 1. \quad (5.16)$$

Assuming that the stability condition (5.14) holds and partitioning $x$ as $x = (x_0, x_1, x_2, ...)$, we obtain

$$x_n = x_1 R^{n-1}, n \geq 1$$

where $R$ is the minimal non negative solution to the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0.$$ 

The two boundary equations involving $x_0$ are

$$x_0 D_0 + x_1 A_{10} = 0,$$

$$x_0 A_{01} + x_1 [A_1 + RA_2] = 0.$$ 

These together with the normalizing condition in (5.16) gives

$$x_1 = x_0 V \text{ where } V = -A_{01}[A_1 + RA_2]^{-1}$$

$$x_0[I + V(I - R)^{-1}]e = 1.$$ 

To see how the system performs, it is instructive to define $y = \sum_{i=1}^{\infty} x_i$. Then $y = (y_1, y_2, y_3)$ where the $y_i$’s indicate status of the customer in service.
5.1.3 System performance measures

1. Probability that system is idle, \( P_{\text{idle}} = x_0 e = 1 - \rho \).

2. Rate of loss of customers, \( R_{\text{loss}} = y_2 S_0^2 = \lambda q(1 - \delta) \).

3. Probability that a customer is lost, \( P_{\text{loss}} = q(1 - \delta) \).

4. Mean number of customers in the system, \( \mu_{NS} = \sum_{i=1}^{\infty} i x_i e \).

5. Mean number of customers in the queue, \( \mu_{NQ} = \sum_{i=2}^{\infty} (i - 1) x_i e \).

6. Probability that the server is serving in required mode, \( P_C = y_1 e + y_3 e = \rho - \lambda q \mu'_2 \).

7. Probability that the server is serving in undesired mode, \( P_I = y_2 e = \lambda q \mu'_2 \).

8. Rate at which customers leave with required service starting in desired service mode, \( R_C = y_1 S_1^0 = \lambda p \).

9. Rate at which customers leave with correct service starting with undesired service, \( R_I = y_3 S_3^0 = \lambda q \delta \).

10. Expected waiting time in the system \( W_S = \frac{\mu_{NS}}{\lambda} \).

5.2 Poisson arrival and phase type service

In this section we analyze the system when arrival follows Poisson process. Service time is phase type distributed as in previous section. Then \( \{(N(t), N^*(t), S(t)), t \geq 0\} \) (see section 5.1) is a continuous time Markov chain with state space \( \{0, 1, 2, \ldots\} \) where \( \mathbb{i} = \{(i, j, k)/1 \leq j \leq 3, 1 \leq k \leq n_j\} \) for \( i \geq 1 \).
Thus the infinitesimal generator is of the form
\[
Q' = \begin{pmatrix}
-\lambda & \lambda \beta \\
S^0 & S - \lambda I & \lambda I \\
S^0 \beta & S - \lambda I & \lambda I \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]

**Theorem 5.2.1.** The system is stable if and only if \( \rho' < 1 \) where
\[
\rho' = \lambda \left[ p\mu'_{1} + q(\mu'_{2} + \delta \mu'_{3}) \right].
\]

**Proof.** From the relation (5.15) we have \( \lambda < \pi S^0 \beta e \) where \( \pi = (\pi_1, \pi_2, \pi_3) \) (with \( \pi_i \)'s as given in (5.10)-(5.12)) is the steady-state probability vector of \( S + S^0 \beta \). The right drift \( \pi S^0 \beta e = \sum_{i=1}^{3} \pi_i S^0_i \).

Multiplying (5.5) by \( e \) on right hand side we get
\[
\sum_{i=1}^{3} \pi_i S^0_i = \frac{c}{q} - \frac{1}{p} \pi_i S^1 e
\]
\[
= \frac{1}{p} \frac{cp}{q} \beta_1 e \text{ from (5.11)}
\]
\[
= \frac{c}{q}
\]
where \( c \) is given in (5.13). Hence the condition for system stability is given by
\[
\lambda < \frac{1}{p\mu'_{1} + q(\mu'_{2} + \delta \mu'_{3})}
\]

\(\square\)

The generator matrix corresponding to the phase changes is \( S + S^0 \beta \) and the stationary probability vector is \( \pi = (\pi_1, \pi_2, \pi_3) \).
Theorem 5.2.2. The steady-state probability vector $x = (x_0, x_1, x_2, \cdots)$ of $Q'$ is given by

$$x_0 = 1 - \rho', \quad x_i = (1 - \rho')\beta R^i, \ i \geq 1,$$

where $R$ is

$$R = \lambda \begin{bmatrix} \lambda I - \lambda p e \beta_1 - S_1 & -\lambda q e \beta_2 & 0 \\ -\lambda p e \beta_1 & \lambda I - \lambda q e \beta_2 - S_2 & -\hat{S}_2 \beta_3 \\ -\lambda p e \beta_1 & -\lambda q e \beta_2 & \lambda I - S_3 \end{bmatrix}^{-1}. \quad (5.17)$$

Proof. Let $x$ be the steady-state probability vector of $Q'$. Then $xQ' = 0$ and $xe = 1$.

The steady-state equations are given by

$$-\lambda x_0 + x_1 S^0 = 0, \quad (5.18)$$
$$\lambda x_0 \beta + x_1 (S - \lambda I) + x_2 S^0 \beta = 0, \quad (5.19)$$
$$\lambda x_{i-1} + x_i (S - \lambda I) + x_{i+1} S^0 \beta = 0, \ i \geq 2. \quad (5.20)$$

From (5.18) we have

$$x_1 S^0 = \lambda x_0. \quad (5.21)$$

Multiplying equations (5.19) and (5.20) by the column vector $e$ on the right hand side leads to

$$x_{i+1} S^0 = \lambda x_i e \text{ for } i \geq 1.$$

Writing $B = e.\beta$ we get $x_{i+1} S^0 \beta = \lambda x_i B$ for $i \geq 1$. Then from (5.19) and (5.20) we obtain

$$x_1 (\lambda I - \lambda B - S) = \lambda x_0 \beta \quad (5.22)$$

and

$$x_i (\lambda I - \lambda B - S) = \lambda x_{i-1}, \text{ for } i \geq 2.$$
Denoting \((\lambda I - \lambda B - S)\) by \(K\), relation (5.22) takes the form \(x_1 = \lambda x_0 \beta K^{-1}\), provided \(K\) is invertible. We now prove the non singularity of \(K\).

Let the vector \(u\) be in the left kernal of \(K\). Then
\[
\lambda u - uS - \lambda(ue)\beta = 0. \tag{5.23}
\]
Suppose \(ue = 0\). Then (5.23) reduces to \(u(\lambda I - S) = 0\). But \((\lambda I - S)\) is nonsingular and hence u = 0.

If \(ue \neq 0\), normalize \(u\) by setting \(ue = 1\). Post multiplying (5.23) by \(e\) gives
\[
uS^0 = 0. \tag{5.24}
\]
Substituting for \(ue\), (5.23) reduces to \(u = \lambda \beta (\lambda I - S)^{-1}\).

From (5.24) we have
\[
\lambda \beta (\lambda I - S)^{-1} S^0 = 0. \tag{5.25}
\]
But \(\beta (\lambda I - S)^{-1} S^0\) is the Laplace-Stieltjes transform at \(s = \lambda > 0\), of the probability distribution \(F(t) = 1 - \beta \exp(St)e\) for \(t \geq 0\). Therefore (5.25) cannot hold and hence \(u = 0\). Thus \(K\) is nonsingular.

The irreducibility of the representation \((\beta, S)\) leads to the irreducibility of the stable \(K\), so that the matrix \(R\) in (5.17) is positive.

We have \(sp(R) < 1\), if \(\rho' < 1\). Therefore the quantity \(x_0\) is given by the normalizing equation
\[
x_0 + x_0 \beta R(I - R)^{-1}e = 1.
\]
Substitution for \(R\) leads to
\[
x_0 - \lambda x_0 \beta (\lambda B + S)^{-1}e = 1. \tag{5.26}
\]
The inverse of \((\lambda B + S)\) is calculated as
\[
(\lambda B + S)^{-1} = S^{-1}(I + \lambda BS^{-1})^{-1} = S^{-1} \sum_{n=0}^{\infty} (-1)^n \lambda^n (BS^{-1})^n
\]
\[
= S^{-1} \left[ I - \lambda \sum_{n=0}^{\infty} (-1)^n \lambda^n (BS^{-1})^n \right] BS^{-1}
\]
\[
= S^{-1} \left[ I - \lambda \sum_{n=0}^{\infty} \rho^n BS^{-1} \right] BS^{-1}
\]
\[
= S^{-1} \left[ I - \lambda (1 - \rho)^{-1} BS^{-1} \right].
\]
From (5.26) we have
\[
x_0 - \lambda x_0 \beta (\lambda B + S)^{-1} e = x_0 - \lambda x_0 \beta \left[ S^{-1} \left( I - \lambda (1 - \rho)^{-1} BS^{-1} \right) \right] e
\]
\[
= x_0 - \lambda x_0 BS^{-1} e + \lambda^2 x_0 (1 - \rho)^{-1} BS^{-1} e
\]
\[
= x_0 + \rho' x_0 + \rho''(1 - \rho') x_0
\]
\[
= (1 - \rho') x_0 = 1,
\]
so that \(x_0 = (1 - \rho')\). \(\square\)

Letting \(y = \sum_{i=1}^{\infty} x_i\), it is obtained that \(y = \rho' \pi\). In the sequel partition \(y = (y_1, y_2, y_3)\), so that \(y_i = \rho' \pi_i\), \(1 \leq i \leq 3\).

### 5.3 Poisson arrival with exponentially distributed service time

In this section we consider customers to arrive according to a Poisson process with rate \(\lambda\) and desired (correct) service time follows exponential distribution with parameter \(\mu\) \((\mu_1' = \mu_3' = \mu)\) and the undesired (incorrect) part of service following phase type distribution with representation \((\beta_2, S_2)\) of order \(n_2\) (see section 5.1). Let \(N(t)\) be the number of customers in the system, \(N^*(t)\) the type
Poisson arrival with exponentially distributed service time of service and $S(t)$ the phase of service at time $t$. $S(t)$ assumes a value between 1 and $n_2$(including both) if server is in undesired phase of service, otherwise 0 or $n_2 + 1$ according as a desired service going on for a customer admitted directly or from undesired state. Then $\{(N(t), N^*(t), S(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{0, 1, 2, \ldots\}$ where

$$i = \{(i, 1, 0), (i, 3, n_2 + 1)\} \cup \{(i, 2, j)/1 \leq j \leq n_2\} \text{ for } i \geq 1.$$ 

Thus the infinitesimal generator is of the form

$$Q = \begin{pmatrix}
0 & 1 & 2 & 3 & \ldots \\
-\lambda & b_0 & & & \\
c_0 & A_1 & A_0 & & \\
& A_2 & A_1 & A_0 & \\
& & \ddots & \ddots & \ddots
\end{pmatrix}$$

where

$$b_0 = \lambda(p, q\beta_2, 0), \quad c_0 = \begin{pmatrix}
\mu \\
\hat{S}_2^0
\end{pmatrix}, A_0 = \lambda I$$

$$A_1 = \begin{pmatrix}
-\lambda - \mu & 0 & 0 \\
0 & S_2 - \lambda I & \hat{S}_2^0 \\
0 & 0 & -\lambda - \mu
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu p & \mu q\beta_2 & 0 \\
p S_2^0 & q S_2^0\beta_2 & 0 \\
\mu p & \mu q\beta_2 & 0
\end{pmatrix}$$

with $S e + S_2^0 + \hat{S}_2^0 = 0$.

### 5.3.1 Stability condition

Consider $A = A_0 + A_1 + A_2$

$$= \begin{pmatrix}
-\mu q & \mu q\beta_2 & 0 \\
p S_2^0 & S_2 + q S_2^0\beta_2 & \hat{S}_2^0 \\
\mu p & \mu q\beta_2 & -\mu
\end{pmatrix}.$$
the generator matrix of the Markov chain corresponding to the phase changes. Let

\[ \Pi = (\pi_0, \hat{\pi}, \pi_{r+1}) \]

be the steady-state probability matrix of \( A \). Solving the relations

\[ \Pi A = 0, \quad \Pi e = 1 \quad (5.27) \]

we obtain

\begin{align*}
-\mu q \pi_0 + p \hat{\pi} S_0^2 + \mu p \pi_{r+1} &= 0 \quad (5.28) \\
\mu q \pi_0 \beta_2 + \hat{\pi} (S_2 + q S_2^0 \beta_2) + \mu q \pi_{r+1} \beta_2 &= 0 \quad (5.29) \\
\hat{\pi} S_0^0 - \mu \pi_{r+1} &= 0. \quad (5.30)
\end{align*}

From equations (5.28) and (5.30),

\[ \mu q \pi_0 = p \left( \hat{\pi} S_2^0 + \hat{\pi} S_2^0 \right) \quad (5.31) \]

This together with (5.29) gives

\[ \hat{\pi} \left( S_2 + S_2^0 \beta_2 + S_2^0 \beta_2 \right) = 0 \]

so that

\[ \hat{\pi} = c \beta_2 (-S_2)^{-1} \quad (5.32) \]

c being a constant and is computed from the normalizing condition. Let \( \delta \) be the probability that a customer getting correct service following one or several incorrect services, and \( \eta \) the probability of staying back in incorrect services. Then

\[ \delta = \beta_2 (-S_2)^{-1} S_2^0 \]

and

\[ \eta = \left( \beta_2 (-S_2)^{-1} e \right)^{-1}. \]

Then the probability that a customer leaves the system without getting required service is

\[ 1 - \delta = \beta_2 (-S_2)^{-1} S_2^0 \]
and the mean time a customer stay back in incorrect services is

\[ \frac{1}{\eta} = (\beta_2(-S_2)^{-1}e) . \]

The normalizing equation is

\[ \pi_0 + \hat{\pi} e + \pi_{r+1} = 1. \]

Substituting for the components of \( \Pi \) which are now computed as

\[ \pi_0 = \frac{pc}{\mu q}, \quad \hat{\pi} e = \frac{c}{\eta}, \quad \pi_{r+1} = \frac{c\delta}{\mu} \]

we get

\[ \frac{pc}{\mu q} + \frac{c}{\eta} + \frac{c\delta}{\mu} = 1 \]

which shows

\[ c = \frac{\mu q \eta}{pq + \mu q + \delta \eta}. \]

**Theorem 5.3.1.** The stability of the system is given by \( \lambda < \frac{1}{q} c. \)

**Proof.** The condition for the stability of the system is \( \Pi A_0 e < \Pi A_2 e. \) Simplification gives \( \Pi A_0 e = \lambda. \) Now \( A_2 e = (\mu, S_0^2, \mu)^T. \) Therefore \( \Pi A_2 e = \mu \pi_0 + \hat{\pi} (S_0^2 + \hat{S}_2^0). \) Substituting for \( \mu \pi_0, \) right hand side becomes \( \frac{1}{q} \hat{\pi} (S_0^2 + \hat{S}_2^0). \) Using equation(5.32) and the fact that \((S_2)^{-1}(S_0^2 + \hat{S}_2^0) = e,\) the result follows. Hence the system is stable if and only if \( \rho < 1, \) where

\[ \rho = \lambda \frac{q}{c}. \quad (5.33) \]

\( \square \)

**5.3.2 Steady-state probability vector**

Let the steady-state probability vector \( \mathbf{x} = (x^*, \mathbf{x}(1), \mathbf{x}(2), \ldots) \) of \( \mathbf{Q} \) be such that \( \mathbf{x} \mathbf{Q} = 0, \mathbf{x} e = 1. \) Partitioning gives \( \mathbf{x}(i) = (x_0(i), \tilde{x}(i), x_{r+1}(i)). \) The relation
\( xQ = 0 \) gives the following system of equations:

\[
-\lambda x^* + x(1)c_0 = 0, \tag{5.34}
\]

\[
x^*b_0 + x(1)A_1 + x(2)A_2 = 0, \tag{5.35}
\]

For \( i \geq 1 \), \( x(i-1)A_0 + x(i)A_1 + x(i+1)A_2 = 0. \tag{5.36} \)

From the matrix geometric structure we obtain

\[
x(i) = x(1)R^{i-1}, \quad i \geq 1
\]

where \( R \) is the minimal non negative solution to the matrix quadratic equation

\[
R^2A_2 + RA_1 + A_0 = 0. \tag{5.37}
\]

Equation (5.34) shows

\[
x^* = \frac{1}{\lambda}x(1)c_0.
\]

Equation (5.35) together with normalizing condition gives

\[
x^*b_0 + x(1)(A_1 + RA_2) = 0
\]

subject to \( x^*e + x(1)(I - R)^{-1}e = 1. \)

Substituting for \( x^* \) we get

\[
x(1) \left( A_1 + RA_2 + \frac{1}{\lambda}c_0b_0 \right) = 0
\]

subject to \( x(1) \left( \frac{1}{\lambda}c_0 + (I - R)^{-1}e \right) = 1. \)

But \( c_0b_0 = \lambda A_2 \) which implies

\[
x(1)(A_1 + RA_2 + A_2) = 0
\]

subject to \( x(1) \left( \frac{1}{\lambda}c_0 + (I - R)^{-1}e \right) = 1. \)
Poisson arrival with exponentially distributed service time

Computation of $R$

$R$ can computed explicitly along the following lines.

We have

$$A_2 = \begin{pmatrix} \mu p & \mu q \beta_2 & 0 \\ p S^0_2 & q S^0_2 \beta_2 & 0 \\ \mu p & \mu q \beta_2 & 0 \end{pmatrix} = \begin{pmatrix} \mu \\ S^0_2 \\ \mu \end{pmatrix} \begin{pmatrix} p & q \beta_2 & 0 \end{pmatrix}$$

so that

$$A_2 e = \begin{pmatrix} \mu \\ S^0_2 \\ \mu \end{pmatrix} = c_0$$

Also from the relation $R A_2 e = A_0 e$, we obtain

$$R A_2 e = \lambda e$$

(5.38)

Now,

$$R^2 A_2 = R^2 \begin{pmatrix} \mu \\ S^0_2 \\ \mu \end{pmatrix} \begin{pmatrix} p & q \beta_2 & 0 \end{pmatrix} = R^2 A_2 e \begin{pmatrix} p & q \beta_2 & 0 \end{pmatrix}$$

Substituting for $R A_2$ from (5.38), we get

$$R^2 A_2 = R \lambda e \begin{pmatrix} p & q \beta_2 & 0 \end{pmatrix}$$

Therefore equation (5.37) gives

$$\lambda R e \begin{pmatrix} p & q \beta_2 & 0 \end{pmatrix} + R A_1 + \lambda I = 0$$

This gives

$$R = \lambda \begin{pmatrix} \mu + \lambda q & -\lambda q \beta_2 & 0 \\ -\lambda p e & \lambda I - \lambda q e \beta_2 - S_2 & -S^0_2 \\ -\lambda p & -\lambda q \beta_2 & \lambda + \mu \end{pmatrix}^{-1}$$
**Lemma 5.3.1.** \( \lambda^* = 1 - \rho \) so that \( x(1) (I - R)^{-1} e = \rho \)

**Proof.** Multiplying by \( e \) on the right side of equation (5.35) and simplifying we get the relation

\[
\lambda x^* + x(1) \begin{pmatrix}
-\lambda - \mu \\
S_2 - \lambda I + S_2^0 \\
-\lambda - \mu
\end{pmatrix} + x(2) \begin{pmatrix}
\mu \\
S_2^0 \\
\mu
\end{pmatrix} = 0. \tag{5.39}
\]

Equation (5.34) gives

\[
\lambda x^* = x(1) \begin{pmatrix}
\mu \\
S_2^0 \\
\mu
\end{pmatrix}. \tag{5.40}
\]

Putting this in (5.39) the following relation is obtained.

\[
x(2) \begin{pmatrix}
\mu \\
S_2^0 \\
\mu
\end{pmatrix} = \lambda x(1) e. \tag{5.41}
\]

Multiplying equation (5.36) on right side by \( e \) and recursive use of the relation results in

\[
x(i) \begin{pmatrix}
\mu \\
S_2^0 \\
\mu
\end{pmatrix} = \lambda x(i - 1) e \quad \text{for } i \geq 3. \tag{5.42}
\]

Adding (5.40), (5.41) and (5.42)

\[
\sum_{i=1}^{\infty} x(i) \begin{pmatrix}
\mu \\
S_2^0 \\
\mu
\end{pmatrix} = \lambda. \tag{5.43}
\]
Adding the system of equations (5.36) with equation (5.35) and using the fact that
\( x^*b_0 = x(1)A_2 \) we get
\[
\sum_{i=1}^{\infty} x(i) A = 0.
\]
But the relation (5.27) says
\[
\sum_{i=1}^{\infty} x(i) = d \Pi \quad \text{for some constant } c
\]
which in turn gives
\[
\sum_{i=1}^{\infty} x(i) = (1 - x^*) \Pi.
\]
Multiplying on the right side by
\[
\begin{pmatrix}
\mu \\
\sl{S_0^2} \\
\mu
\end{pmatrix}
\]
and using the relation in (5.33)
\[
\sum_{i=1}^{\infty} x(i) \begin{pmatrix}
\mu \\
\sl{S_0^2} \\
\mu
\end{pmatrix} = (1 - x^*) \frac{\lambda}{\rho}.
\]
(5.44)
The result follows from (5.43) and (5.44).

5.3.3 System performance measures

1. Probability that the system is idle, \( P_0 = x^* \).

2. Rate of loss, \( R_{\text{loss}} = \sum_{i=1}^{\infty} \bar{x}(i) S_0^2 = \lambda q(1 - \delta) \).

3. Probability of loss, \( P_{\text{loss}} = q(1 - \delta) \).

4. Mean number of customers in the system,
\[
\mu_{\text{ns}} = \sum_{i=1}^{\infty} ix(i)e = x(1)(1 - R)^{-2}e.
\]
5. Mean number of customers in the queue,
\[ \mu_{nq} = \sum_{i=1}^{\infty} (i - 1) \mathbf{x}(i) \mathbf{e} = \mathbf{x}(1) (I - R)^{-2} \mathbf{e} - \mathbf{x}(1) (I - R)^{-1} \mathbf{e}. \]

6. Probability that the server is busy serving in correct mode,
\[ P_c = \sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \rho (\pi_0 + \pi_{r+1}) = \rho - \frac{\lambda q}{\eta}. \]

7. Probability that the server is busy serving in incorrect mode,
\[ P_i = \sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} 0 \\ \mathbf{e} \\ 0 \end{pmatrix} = \rho \pi \mathbf{e} = \frac{\lambda q}{\eta}. \]

5.4 An illustration

In this section we consider a queueing model consisting of two service stations—preliminary service and main service. Customers arrive to this system according to a MAP (Markovian Arrival Process) with representation \((D_0, D_1)\) of order \(m\). A customer, who is taken for service is directly selected for main service with probability \(p\) or to the preliminary service with probability \(q = 1 - p\). A threshold clock starts ticking if a customer enters to preliminary service. When the duration of preliminary service exceeds the threshold clock, the customer moves out of the system, else he goes to main service. The threshold clock follows exponential distribution with parameter \(\zeta\). Service time of the customers at these stations follow phase type distributions with representation \((\alpha, S_P), (\gamma, S_M)\) and of order \(a, b\) respectively. Write \(S_P^0 + \zeta \mathbf{e} = -S_P \mathbf{e}\) and \(S_M^0 = -S_M \mathbf{e}\) where \(\mathbf{e}\) is a column vector of 1’s of appropriate order. Hence service time of a customer can be modeled as a phase type distribution with representation \((\xi, U)\) of order \(a + 2b\) such that \(U \mathbf{e} + U^0 = 0\) where
\[ \xi = \begin{pmatrix} p \gamma & q \alpha & 0 \end{pmatrix} \]
An illustration

\[
U = \begin{pmatrix}
S_M & 0 & 0 \\
0 & S_P & S_P^0 \gamma \\
0 & 0 & S_M
\end{pmatrix}, \quad U^0 = \begin{pmatrix}
S_M^0 \\
\zeta e \\
S_M^0
\end{pmatrix}.
\]

Let \(N(t), N^*(t), S(t), A(t)\) denote respectively the number of customers in the system, nature of service, phase of service and phase of arrival at time \(t\) with

\[
N^*(t) = \begin{cases}
1 & \text{main service} \\
2 & \text{preliminary service} \\
3 & \text{one that come from preliminary service}
\end{cases}
\]

The process \(\Omega = \{(N(T), N^*(t), S(t), A(t)), t \geq 0\}\) is a continuous time Markov chain with state space \(\{(n, i, j, k)/i = 1, 3; 1 \leq j \leq b, 1 \leq k \leq m\} \cup \{(n, 2, j, k)/1 \leq j \leq a, 1 \leq k \leq m\}\) for \(n \geq 1\).

Note that when \(N(t) = 0\), the only other component in the state vector is \(A(t)\).

Thus the infinitesimal generator of \(\Omega\) is of the form

\[
Q^* = \begin{pmatrix}
D_0 & A_{01} & A_0 \\
A_{10} & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
& & & \ddots & \ddots & \ddots
\end{pmatrix}
\]

where \(A_{01} = \xi \otimes D_1, A_{10} = U^0 \otimes I_m, A_0 = I_{n+2b} \otimes D_1, A_1 = U \oplus D_0, A_2 = U^0 \xi \otimes I_m\).

The infinitesimal generator \(Q^*\) is of the same form as \(Q\) of the model described initially. Thus the analysis of the Markov chain with infinitesimal generator \(Q^*\) can be done in the same way as for \(Q\).

The significance of this model is as follows: customer arriving to a single server belong to two categories, though they join the same waiting line. While taking for service the category will be decided. Call them category 1 and category 2, respectively. Category 1 are qualified for the main service without undergoing preliminary service. However, category 2 have to be given the preliminary service before admitted to main service. However, if such customers do not get service in
preliminary before realization of the timer (random clock), they get disqualified and so leave the system forever. On the other hand those among category 2, completing service successfully before timer realization in preliminary, are immediately admitted to main service. On completion of that service such customers leave the system.

Remark 5.4.1. In telecommunication it is this type of situation that is often encountered. Packages have to identify the server in idle state; then wait for a while. But in the mean time another message may get through, making the server busy. Then the customer (packet) under consideration has to go through a series of contention windows. These passages could be regarded as unwanted service. In case the process of going through contention windows exceeds a threshold time limit (time out/ clock realization), the message will not get served.

Remark 5.4.2. The problem discussed in Madan [46] and Medhi [48] could be arrived at from our model as follows. Suppose that we reverse the order of preliminary and main service, that is, main service first and preliminary (hereafter we call the second as optional) service next. Then after completion of main service, the customer asks for an optional service with probability $1 - q$ (this optional service time has exponential distribution in Madan [46]). This could be regarded as an instantaneous feedback as head of waiting line and get served according to a different distribution. With probability $q$, the customer leaves the system immediately after main service completion.

5.5 Numerical illustration

The following numerical illustration is based on the description in Section 2. We fix parameters $n_1 = 2, n_2 = 3, n_3 = 4, \beta_1 = (0.4, 0.6)$,
\[
\beta_2 = (0.3 \ 0.5 \ 0.2), \beta_3 = (0.2 \ 0.3 \ 0.3 \ 0.2), \\
S_1 = \begin{bmatrix} * & 6 \\ 8 & * \end{bmatrix}, S_1^0 = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \text{ with } S_1 \mathbf{e} + S_1^0 = 0,
\]
\[
S_2 = \begin{bmatrix} * & 5 & 5 \\ 6 & * & 6 \\ 5 & 7 & * \end{bmatrix}, S_2^0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, S_2^0 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \text{ with } S_2 \mathbf{e} + S_2^0 + \hat{S}_2^0 = 0,
\]
\[
S_3 = \begin{bmatrix} * & 7 & 8 & 9 \\ 6 & * & 7 & 7 \\ 6 & 6 & * \\ 8 & 7 & 6 & * \end{bmatrix}, S_3^0 = \begin{bmatrix} 6 \\ 8 \\ 9 \end{bmatrix} \text{ with } S_3 \mathbf{e} + S_3^0 = 0.
\]

For the arrival process, we consider the following two sets of values for \(D_0\) and \(D_1\) as follows. The arrival processes labeled \(MNCA\) and \(MPCA\) respectively, have negative and positive correlation for two successive inter-arrival time with values -0.48891 and 0.48891. The standard deviation of the inter-arrival time of these two arrival processes are, respectively, 0.2819 and 0.2819.

1. **MAP with negative correlation (MNCA):**

\[
D_0 = \begin{bmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.05011 & 0 & 4.96099 \\ 1117.4625 & 0 & 11.2875 \end{bmatrix}
\]

2. **MAP with positive correlation (MPCA):**

\[
D_0 = \begin{bmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 4.96099 & 0 & 0.05011 \\ 11.2875 & 0 & 1117.4625 \end{bmatrix}
\]

The output in Tables 5.1 and 5.2 are on expected lines. Note that \(P_{loss}\) decreases with increasing value of \(p\). The value of \(P_C(R_C)\) steadily increases with \(p\) and values of \(P_I(R_I)\) and \(W_S\) decrease with increase in value of \(p\), as expected.
Single server queue with several services

<table>
<thead>
<tr>
<th>$p$</th>
<th>$P_{\text{loss}}$</th>
<th>$\mu_{NS}$</th>
<th>$P_C$</th>
<th>$P_I$</th>
<th>$R_C$</th>
<th>$R_I$</th>
<th>$W_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.2136</td>
<td>7.5229</td>
<td>0.5242</td>
<td>0.3921</td>
<td>2</td>
<td>1.9320</td>
<td>1.5046</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1780</td>
<td>4.9744</td>
<td>0.5483</td>
<td>0.3267</td>
<td>2.5</td>
<td>1.6100</td>
<td>0.9949</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1424</td>
<td>3.6690</td>
<td>0.5724</td>
<td>0.2614</td>
<td>3</td>
<td>1.2880</td>
<td>0.7338</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1068</td>
<td>2.8654</td>
<td>0.5965</td>
<td>0.1960</td>
<td>3.5</td>
<td>0.9660</td>
<td>0.5731</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0712</td>
<td>2.3138</td>
<td>0.6206</td>
<td>0.1307</td>
<td>4</td>
<td>0.6440</td>
<td>0.4628</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0356</td>
<td>1.9069</td>
<td>0.6447</td>
<td>0.0653</td>
<td>4.5</td>
<td>0.3220</td>
<td>0.3814</td>
</tr>
</tbody>
</table>

Table 5.1: Effect of $p$ for MNCA

<table>
<thead>
<tr>
<th>$p$</th>
<th>$P_{\text{loss}}$</th>
<th>$\mu_{NS}$</th>
<th>$P_C$</th>
<th>$P_I$</th>
<th>$R_C$</th>
<th>$R_I$</th>
<th>$W_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.2136</td>
<td>546.8179</td>
<td>0.5242</td>
<td>0.3921</td>
<td>2</td>
<td>1.9320</td>
<td>109.3646</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1780</td>
<td>349.9587</td>
<td>0.5483</td>
<td>0.3267</td>
<td>2.5</td>
<td>1.6100</td>
<td>69.9924</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1424</td>
<td>250.7699</td>
<td>0.5724</td>
<td>0.2614</td>
<td>3</td>
<td>1.2880</td>
<td>50.1545</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1068</td>
<td>191.0008</td>
<td>0.5965</td>
<td>0.1960</td>
<td>3.5</td>
<td>0.9660</td>
<td>38.2005</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0712</td>
<td>151.0402</td>
<td>0.6206</td>
<td>0.1307</td>
<td>4</td>
<td>0.6440</td>
<td>30.2083</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0356</td>
<td>122.4351</td>
<td>0.6446</td>
<td>0.0653</td>
<td>4.5</td>
<td>0.3220</td>
<td>24.4873</td>
</tr>
</tbody>
</table>

Table 5.2: Effect of $p$ for MP CA

The main comparison in Tables 5.1 and 5.2 is between values of $\mu_{NS}$ in MNCA and MP CA. Both decrease with increase in value of $p$. However, MNCA has much smaller values compared to their MP CA counterparts. This indicates that positive correlation in the arrival process results in accumulation of large number of customers in the system.

5.6 $M/G/1$ Model

In this section we consider an $M/G/1$ system with two service stations – preliminary service and main service. Customers arrive to this system according to
a Poisson process with rate $\lambda$. A customer, when taken for service, is directly selected for main service with probability $p$ or to the preliminary service with probability $q (= 1 - p)$. A threshold clock starts ticking if a customer enters to preliminary service. When the duration of preliminary service exceeds the threshold clock, the customer moves out of the system, else he goes to main service. The threshold clock follows exponential distribution with parameter $\zeta$. Here the service time, $V_p, V_m$ of the preliminary and main services are independent having general distributions with distribution function $G_1(\cdot), G_2(\cdot)$, LST $G^*_1(\cdot), G^*_2(\cdot)$ respectively.

The (total) service time $V$ of a unit is

$$V = \begin{cases} V_f & \text{with probability } q \cdot P(G_1(\cdot) > \exp(\zeta)) \\ V_p & \text{with probability } q \cdot P(G_1(\cdot) < \exp(\zeta)) \\ V_m & \text{with probability } p \end{cases}$$

where $V_f$ is the duration of threshold clock realization. Thus

$$G(t) = P(V \leq t) = q \left[ \int_0^t \zeta e^{-\zeta u}(1 - G_1(u))du + \int_0^t e^{-\zeta u}G_1(u)dG_2(t - u) \right] + p \int_0^t dG_2(u)$$

The LST $G^*(s)$ of $V$ is given by

$$G^*(s) = \int_0^\infty e^{-st}dG(t).$$

**Remark 5.6.1.** This modelling closely resembles the protocol IEEE 802.11. This is so because of a message generated has to wait before checking for idle server; if server is busy it has to go through a series of contention windows and then look for idle server. In case this process takes longer duration than the life of message (before its significance is lost), then the message does not serve any purpose. In the opposite case it is transmitted before its expiry time.
Remark 5.6.2. Assume the random clock to be of infinite duration (i.e., its rate of realization goes to zero). Now interchange the roles of preliminary and main services (in this case, we call the preliminary service, which is the second one now, as optional service). Invariably main service is given for all customers. Thus the main service is followed by an optional service to which customers, on completion of main service, proceed with probability \( q \). Then our model reduces to Madan [46] with exponentially distributed optional service and to Medhi [48] in the case of arbitrarily distributed optional service time.

Transient solution

The supplementary variable technique (see Cox [19], Medhi [47]) could be used to get the transient solution. Suppose that the general distribution \( G(x) = P(V \leq x) \) has the hazard function \( h(x) = \frac{dG(x)}{1 - G(x)} \) and the probability density function of \( V \) is given by

\[
g(x) = h(x) \exp\{-N(x)\}
\]

where

\[
N(x) = \int_0^x h(u)du; \quad N(0) = 0 \quad \text{and} \quad \frac{d}{dx}N(x) = h(x).
\]

If \( V \) is the total service time, then \( h(x)dx = P(\text{service will be completed in } (x, x + dx) \text{ given that service time exceeds } x) \) and \( E(V) = \int xg(x)dx = -G^{*'(1)}(0) \).
The supplementary variable $X(t)$ considered is defined below. Let

- $N(t)$ = system size at time $t$
- $X(t)$ = time already spent in service up to $t$ of a unit receiving service
- $p_n(t) = P(N(t) = n)$ with $p_0(0) = 1$
- $p_n(t, x)dx = P(N(t) = n, x \leq X(t) < x + dx)$, $n \geq 1$

- $p_n(t) = \int_0^\infty p_n(t, x)dx$
- $Q(t, z) = \sum_{n=0}^\infty p_n(t)z^n$
- $Q(t, x, z) = \sum_{n=1}^\infty p_n(t, x)z^n$

Now we have

$$p_0(t + \delta t) = [1 - \lambda \delta t + o(\delta t)]p_0(t) + \int_0^\infty p_1(t, x)h(x)d\delta t.$$  

As $\delta t \to 0$,

$$\frac{\partial}{\partial t}p_0(t) = -\lambda p_0(t) + \int_0^\infty p_1(t, x)h(x)dx.$$  

(5.45)

For $\delta x > 0$,

$$p_1(t + \delta t, x + \delta x) = [1 - \lambda \delta t + o(\delta t)][1 - h(x)\delta x + o(\delta x)]p_1(t, x).$$

Subtracting and adding a term $p_1(t, x + \delta x)$ to the LHS, then dividing by $\delta t(\delta x)$ and taking as $\delta t \to 0(\delta x \to 0)$, we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)p_1(t, x) = -(\lambda + h(x))p_1(t, x).$$  

(5.46)
Single server queue with several services

For $n \geq 0$,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) p_n(t, x) = -(\lambda + h(x))p_n(t, x) + \lambda p_{n-1}(t, x). \quad (5.47)$$

We have the following boundary conditions:

$$p_1(t, 0) = \int_0^\infty p_2(t, x)h(x)dx + \lambda p_0(t) \quad (5.48)$$

and

$$p_n(t, 0) = \int_0^\infty p_{n+1}(t, x)h(x)dx, \quad n \geq 2. \quad (5.49)$$

Multiplying (5.47) by $z^n$, $n = 2, 3, \ldots$ and (5.46) by $z$, then adding all the terms we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \sum_{n=1}^\infty p_n(t, x)z^n = -(\lambda + h(x))\sum_{n=1}^\infty p_n(t, x) + \lambda \sum_{n=2}^\infty p_{n-1}(t, x)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) Q(t, x, z) = -(\lambda - \lambda z + h(x))Q(t, x, z). \quad (5.50)$$

Now multiplying (5.49) by $z^n$, $n = 2, 3, \ldots$ and (5.48) by $z$, then adding the terms we have

$$Q(t, 0, z) = \int_0^\infty \left(\sum_{n=1}^\infty p_{n+1}(t, x)z^n\right) h(x)dx + \lambda z p_0(t). \quad (5.51)$$

Now

$$\int_0^\infty \left(\sum_{n=1}^\infty p_{n+1}(t, x)z^n\right) h(x)dx$$
\[ M/G/1 \text{ Model} \]

\[
= \int_{0}^{\infty} \left( \frac{1}{z} \right) \sum_{n=1}^{\infty} p_{n+1}(t, x)z^{n+1}h(x)dx
\]

\[
= \int_{0}^{\infty} \left( \frac{1}{z} \right) \left[ \sum_{n=1}^{\infty} p_{n}(t, x)z^{n} - p_{1}(t, x)z \right] h(x)dx
\]

\[
= \left( \frac{1}{z} \right) \int_{0}^{\infty} \left[ Q(t, x, z) - p_{1}(t, x)z \right] h(x)dx
\]

\[
= \left( \frac{1}{z} \right) \left[ \int_{0}^{\infty} Q(t, x, z)h(x)dx - z(p'_{0}(t) + \lambda p_{0}(t)) \right] \text{ by (5.45)}
\]

Thus (5.51) reduces to

\[
Q(t, 0, z) = \left( \frac{1}{z} \right) \left[ \int_{0}^{\infty} Q(t, x, z)h(x)dx - z(p'_{0}(t) + \lambda p_{0}(t)) \right] + \lambda zp_{0}(t)
\]

\[
= \left( \frac{1}{z} \right) \left[ \int_{0}^{\infty} Q(t, x, z)h(x)dx - z(p'_{0}(t) + \lambda p_{0}(t)) + \lambda z^{2}p_{0}(t) \right]
\]

\[
zQ(t, 0, z) = \int_{0}^{\infty} Q(t, x, z)h(x)dx - zp'_{0}(t) + \lambda z(z - 1)p_{0}(t). \quad (5.52)
\]

The partial differential equation (5.50) can be solved using the boundary condition (5.52) and the normalizing condition \( \sum_{n=0}^{\infty} p_{n}(t) = 1. \)

**Steady-state distribution**

Let

\[
\lim_{t \to \infty} p_{n}(t) = p_{n}, \quad n \geq 0
\]

and

\[
\lim_{t \to \infty} p_{n}(t, x) = p_{n}(x), \quad x > 0, n \geq 1
\]

\[
= p_{0}(x) = 0, \quad x > 0.
\]
Then \( \{p_n, n \geq 0\} \) gives the distribution of the general time system size. Let
\[
Q(x, z) = \sum_{n=1}^{\infty} p_n(x) z^n
\]
\[
= \sum_{n=1}^{\infty} \left[ \lim_{t \to \infty} p_n(t, x) \right] z^n
\]
\[
= \lim_{t \to \infty} \left[ \sum_{n=1}^{\infty} p_n(t, x) z^n \right]
\]
\[
= \lim_{t \to \infty} Q(t, x, z)
\]
and
\[
Q(z) = \int_0^{\infty} Q(x, z) dx.
\]
Then
\[
(5.45) \Rightarrow \lambda p_0 = \int_0^{\infty} p_1(x) h(x) dx
\]
\[
(5.46) \text{ and } (5.47) \Rightarrow \frac{\partial}{\partial x} p_n(x) = -(\lambda + h(x)) p_n(x) + \lambda p_{n-1}(x), \ n \geq 1
\]
\[
(5.48) \Rightarrow p_1(0) = \int_0^{\infty} p_2(x) h(x) dx + \lambda p_0
\]
\[
(5.49) \Rightarrow p_n(0) = \int_0^{\infty} p_{n+1}(x) h(x) dx, \ n \geq 2.
\]
The partial differential equation (5.50) and the boundary condition (5.52) reduces to
\[
\frac{d}{dx} Q(x, z) = -(\lambda - \lambda z + h(x)) Q(x, z)
\]  
(5.53)
\[
z Q(0, z) = \int_0^{\infty} Q(x, z) h(x) dx + \lambda z(z - 1) p_0
\]  
(5.54)
and
\[
p_0 + Q(1) = 1.
\]  
(5.55)
From relation (5.53)
\[ \int \frac{dQ(x, z)}{Q(x, z)} = \int -\lambda - \lambda z + h(x) \, dx \]

\[ \log(Q(x, z)) = \log c (-\lambda(1 - z)x - N(x)) \]
\[ Q(x, z) = c \exp(-\lambda(1 - z)x - N(x)) \]
\[ Q(0, z) = Q \exp(-\lambda(1 - z)x - N(x)) \] (5.56)

Substituting (5.56) in (5.54) we get
\[ zQ(0, z) = \int_{0}^{\infty} Q(x, z) e^{(-\lambda(1-z)x-N(x))} h(x) \, dx + \lambda z(z - 1)p_0 \]

\[ = Q(0, z) \int_{0}^{\infty} e^{-\lambda(1-z)x} \left[ e^{-N(x)} h(x) \right] \, dx + \lambda z(z - 1)p_0 \]

\[ = Q(0, z) G^*(\lambda(1 - z)) + \lambda z(z - 1)p_0. \]

Thus
\[ Q(0, z) = \frac{\lambda z(z - 1)p_0}{z - G^*(\lambda - \lambda z)}. \] (5.57)

Now from (5.56) we have
\[ Q(z) = \int_{0}^{\infty} Q(x, z) \, dx \]

\[ = \int_{0}^{\infty} Q(0, z) e^{(-\lambda(1-z)x-N(x))} \, dx \]

\[ = Q(0, z) \int_{0}^{\infty} e^{(-\lambda(1-z)x)} \, e^{-N(x)} \, dx \]

\[ = \frac{Q(0, z)}{\lambda(1 - z)} \left[ 1 - \int_{0}^{\infty} e^{-\lambda(1-z)x} \left( e^{-N(x)} h(x) \right) \, dx \right] \]
\[ Q(z) = \frac{Q(0, z)}{\lambda(1 - z)} [1 - G^*(\lambda - \lambda z)] \]

From this and equation (5.57) we get

\[ Q(z) = \frac{z[G^*(\lambda - \lambda z) - 1]p_0}{z - G^*(\lambda - \lambda z)} \]

Using L'Hospital rule, we get

\[
Q(1) = \lim_{z \to 1} Q(z)
= p_0 \frac{[G^*(\lambda - \lambda z) - 1] + z\lambda G^*(1)(\lambda - \lambda z)}{1 + \lambda G^*(1)(\lambda - \lambda z)}
= p_0 \frac{\lambda E(V)}{1 - \lambda E(V)}
\]

From (5.55) we obtain

\[ p_0 = 1 - \lambda E(V). \]

Hence

\[ Q(z) = \frac{z[G^*(\lambda - \lambda z) - 1][1 - \lambda E(V)]}{z - G^*(\lambda - \lambda z)}. \]

**Busy period**

Let \( T \) be the length of a busy period (starting with a customer arrival to an idle server, until the becomes idle again). Define \( B(t) = P(T \leq t) \). Then \( B(t) \) satisfies the relation

\[
B(t) = \int_0^t \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} B^{*k}(t - u) dG(u) \quad (5.58)
\]
The Laplace Stieltjes Transform (LST) of busy period $B(t)$ be denoted by $B^*(s)$. That is,

$$B^*(s) = \int_0^\infty e^{-st} dB(t) \quad (\text{for } \text{Re}(s) > 0)$$

$$= \int_0^\infty e^{-st} \int_0^t \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} B^*(t-u) dG(u) dt$$

$$= \int_0^\infty \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-su} \int_u^\infty e^{-s(t-u)} B^*(t-u) dtdG(u)$$

$$= \int_0^\infty \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-sB^*(s)} dG(u)$$

$$= \int_0^\infty \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-(\lambda + s)u} dG(u)$$

$$= \int_0^\infty e^{-(\lambda + s - \lambda B^*(s))u} dG(u)$$

Therefore

$$B^*(s) = G^*(\lambda + s - \lambda B^*(s)).$$

From this the mean and higher moments of the number of customers in the system can be computed.