Chapter 6

A MAP/PH/1 Queue with Uncertainty in Selection of Type of Service

In the previous chapter we assumed that the server offered $n$ distinct services, of which only one was the needed/desired service for each customer. However, due to certain complex situation neither the server nor the customer is aware of the exact needed service. The rest of the services may turn out to be harmful/ineffective. A typical example is the Chikungunya, the symptoms of which varied from person to person. Accordingly physicians prescribed medicines to the patients; however, those who did not receive the right medication within a specified time were rendered physically/mentally handicapped. In this chapter we extend the model described in chapter 5 to the case of $n(n > 1)$ distinct services offered by a server with distinct customers requiring any one among the $n$ services which

Part of this chapter is included in the following paper.

we label as the desired service for that customer. The desired service may vary from customer to customer. For example, a customer dialing a customer care center for a specific service.

We analyze a single server system providing \( n \) distinct services; customer arrival follows a Markovian arrival process. At the time when taken for service the service requirement is correctly diagnosed with probability \( \theta \); with complementary probability \( (1 - \theta) \) the identification goes wrong. As a consequence of correct diagnosis, service in correct mode immediately starts, whose duration has exponential distribution with parameter \( \mu_i \) if service required is in state \( i \) and the customer leaves the system after service. However, if initially the customer is admitted to one of the incorrect services (with probability \( p_i \), it is diagnosed as requiring type \( i \) service, \( i = 1, 2, \ldots, n \)), it may stay in this class, moving from one incorrect to another incorrect, until finally all turn out to be failure and the customer turns out to be unfit for further service. It may also happen that at some stage of service in incorrect class, the service provider identifies that the customer is being served in the incorrect set of services and so immediately takes him to the actually required service stage. At this point, the customer starts required service and leaves the system on completion of service. But then how long is it possible to stay in service in the incorrect set of states? We assume that a timer with exponentially distributed duration starts ticking the moment a customer starts getting his service. If correct diagnosis is made of the desired service during its sojourn in the incorrect set of states before this random clock (timer) realizes, then the customer is immediately transferred for service to the correct state. On completion of service, assumed exponentially distributed with parameter \( \mu_i \), the customer leaves the system. On the other hand if the timer realizes before the customer’s service need is correctly diagnosed, then no further service is provided to that customer since it is rendered useless as a consequence of service in the incorrect set of states.

In Section 1, the mathematical model is described. Section 2 provides the
steady-state analysis and some performance measures including the expected service time of a customer. Effect of various parameters on performance measures of the system are numerically computed in Section 3.

6.1 Mathematical formulation

The assumptions leading to the formulation of the mathematical model are

- Arrival of customers to the system is according to the MAP. We use the same notations used in the previous chapter associated with MAP.

- The probability that a customer gets desired (correct) service from the very beginning is $\theta$; denote $1 - \theta = \hat{\theta}$.

- The probability that a customer requires the $i^{th}$ type of service is $p_i$ so that $p_1 + p_2 + \ldots + p_n = 1$.

- If $i$ is the required type of service for a customer and service starts correctly then corresponding service time is exponentially distributed with mean service rate $\mu_i, i = 1, 2, \ldots, n$.

- If $i$ is the required type of service for a customer write $\beta^{(i)} = \left(\beta^{(i)}_k\right)_{1 \leq k \leq n, k \neq i}$ where $\beta^{(i)}_k$ is the probability that a customer in need of $i^{th}$ service starts with $k^{th}(k \neq i)$. The rate of transition to $j^{th}$ state of incorrect service after completing service in $k^{th}$ state is $\mu_{kj}$ for, $k \neq i; j \in \{1, 2, \ldots, n\}, j \neq i$.

- A random threshold clock(timer) which follows exponential distribution with mean rate $\gamma$ starts ticking from the very beginning of service to the specific customer so that the customer is pushed out of the system if the clock expires before service completion in undesired service.
The last two assumptions indicate that only if the service requirement is correctly diagnosed right at beginning when taken for service, does the customer have an exponentially distributed service time. In the other case the service time turns out to be phase type distributed (initially in state(s) which are not the correct one and then get absorbed due to realization of timer or in the absence of realization of timer during service in the undesired states, thus escaping to the correct state of service, where there is additional exponentially distributed service requirement).

Let \( N_1(t) \) be the number of customers in the system, \( N_2(t) \) and \( N_4(t) \) respectively the required service type and the type of service being provided and \( N_3(t) \) the mode of service whether desired from the very beginning of service, unwanted or moved from undesired to desired, designated by 1, 2 and 3 respectively and \( A(t) \) be the arrival phase at time \( t \).

Then, \( \{(N_1(t), N_2(t), N_3(t), N_4(t), A(t)), t \geq 0\} \) is a CTMC with state space \( \Omega = \{(0, a)/1 \leq a \leq m\} \cup \{(i, j, k, \ell, a)/i \in Z^+, 1 \leq j \leq n, \ell = j; k = 1, 3; 1 \leq a \leq m\} \cup \{(i, j, 2, \ell, a)/i \in Z^+, 1 \leq j, \ell \leq n; \ell \neq j; 1 \leq a \leq m\} \). The infinitesimal generator \( Q \) of this CTMC is a LIQBD where

\[
Q = \begin{pmatrix}
B_{00} & B_{01} & & \\
B_{10} & B_1 & B_0 & \\
& B_2 & B_1 & B_0 \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{pmatrix}.
\]

In the above matrix \( B_{00} = D_0, \ B_{01} = \hat{\alpha} \otimes D_1, \ B_{10} = C \otimes I_m, \ B_0 = I_{n(n+1)} \otimes D_1, B_1 = T \oplus D_0, \ B_2 = H \otimes I_m \). Here \( \hat{\alpha} = \left( p_1 \alpha_1, \ p_2 \alpha_2, \ \cdots, \ p_n \alpha_n \right) \) with \( \alpha_i = \left( \theta, \ \hat{\beta}^{(i)}, \ 0 \right), \ 1 \leq i \leq n, \) and \( C = \left( c^{(1)}, \ c^{(2)}, \ \cdots, \ c^{(n)} \right)^T \).

Let \( \Delta_1^{(i)} = \begin{pmatrix} \mu_i \\ 0 \\ \mu_i \end{pmatrix}, \) and \( \Delta_2 = \begin{pmatrix} 0 \\ \gamma e \\ 0 \end{pmatrix} \). Then \( c^{(i)} = \Delta_1^{(i)} + \Delta_2. \)
Mathematical formulation

\[ H = [M_{ij}] \text{ where } M_{ij} = \left( \begin{array}{ccc} \mu_{i} \mu_{j} \theta & \mu_{i} \mu_{j} \theta \beta^{(j)} & 0 \\ U_{0} \mu_{i} \theta & U_{0} \mu_{j} \theta \beta^{(j)} & 0 \\ \mu_{i} \mu_{j} \theta & \mu_{i} \mu_{j} \theta \beta^{(j)} & 0 \end{array} \right), \quad 1 \leq i, j \leq n. \]

\[ T = \text{diag}(T_1, T_2, ..., T_n), \quad T_i = \left( \begin{array}{ccc} -\mu_i & 0 & 0 \\ 0 & S^{(i)} & S^{(i)}_0 \\ 0 & 0 & -\mu_i \end{array} \right), \quad 1 \leq i \leq n \]

with

\[ S^{(i)} = \left( \begin{array}{cccccccc} 1 & 2 & i-1 & i+1 & n \\ 2 & \mu^{(i)}_1 & \mu^{(i)}_{12} & \mu^{(i)}_{1(i-1)} & \mu^{(i)}_{1(i+1)} & \cdots & \mu^{(i)}_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i-1 & \mu^{(i)}_{(i-1)1} & \mu^{(i)}_{(i-1)2} & \mu^{(i)}_{1(i-1)} & \mu^{(i)}_{1(i+1)} & \cdots & \mu^{(i)}_{1n} \\ i+1 & \mu^{(i)}_{(i+1)1} & \mu^{(i)}_{(i+1)2} & \mu^{(i)}_{(i+1)(i-1)} & \mu^{(i)}_{(i+1)(i+1)} & \cdots & \mu^{(i)}_{(i+1)n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n & \mu_{n1} & \mu_{n2} & \cdots & \mu^{(i)}_{(i-1)} & \cdots & \mu^{(i)}_{ni} \end{array} \right). \]

Here, \( \mu^{(i)}_k = -\left( \sum_{j=1}^{n} \mu_{kj} + \gamma \right), \quad k = 1, 2, ..., i-1, i+1, ..., n \) and

\[ S^{(i)}_0 = \left( \begin{array}{cccc} \mu_{1i} & \mu_{2i} & \cdots & \mu_{(i-1)i} \\ \mu_{1i} & \mu_{2i} & \cdots & \mu_{(i+1)i} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{1i} & \mu_{2i} & \cdots & \mu_{ni} \end{array} \right)^T. \]
6.2 Steady-state analysis

We proceed with the steady-state analysis of the queueing system under study. Naturally we have to look for the condition for stability.

6.2.1 Stability condition

We consider the matrix \( B = B_0 + B_1 + B_2 \) representing the phase changes for determining the stability condition of the original system. We have \( B = (T + H) \oplus D \)

where \( T + H = (E_{ij})_{1 \leq i,j \leq n}, E_{ij} = M_{ij} \) for \( i \neq j \)

\[
E_{ii} = \begin{pmatrix}
-\mu_i(1 - p_i \theta) & \mu_i p_i \beta^{(i)}_0 & 0 \\
U^0 p_i \theta & U^0 p_i \hat{\theta} \beta^{(i)} & S_i^0 \\
\mu_i p_i \theta & \mu_i p_i \hat{\theta} \beta^{(i)} & -\mu_i
\end{pmatrix},
\]

Let \( \tilde{\pi} = (\hat{\pi}_1, \hat{\pi}_2, ..., \hat{\pi}_n) \) be the stationary probability vector of the Markov chain corresponding to the generator \( T + H \) and \( \eta \) be that of \( D \). Then

\[
\tilde{\pi}(T + H) = 0, \quad \tilde{\pi} e = 1.
\]

\[
\eta D = 0, \quad \eta e = 1.
\]

Thus the stationary probability vector of \( B \) is \( \Pi = \tilde{\pi} \otimes \eta \).

An algorithm for computing \( \tilde{\pi} \) is given below:

\[
\hat{\pi}_{n-k} = \sum_{i=1}^{n-k-1} \hat{\pi}_i F_i(n-k) \quad ; 0 \leq k \leq n - 2
\]

For \( 1 \leq i \leq n - 1 \),
Steady-state analysis

\[ F_{i(n-k)} = \begin{cases} \frac{-B_{in}E^{-1}_{tn}}{n-k}, & k = 0, \\ \frac{-(B_{i(n-k)} + \sum_{m=1}^{k} U_{im}) (E_{(n-k)(n-k)} + \sum_{m=1}^{k} U_{(n-k)m})^{-1}}{1 \leq k \leq n - 2}, \end{cases} \]

with

\[ U_{im} = \sum_{J_{r-1+1} \leq J_{r} \leq n-m+r} F_{r_1} F_{r_1, r_2} \cdots F_{r_{m-1}, r_m} B_{J_0(n-k)}. \]

\[ \hat{\pi}_1 \] is obtained from

\[ \hat{\pi}_1 \left(I + \sum_{m=1}^{n-1} V_{m}\right) e = 1, \]

where

\[ V_{m} = \sum_{r_{j-1+1} \leq r_j \leq n-(m-j)} F_{r_1} \prod_{j=1}^{m-1} F_{r_j, r_{j+1}}. \]

The LIQBD description of the model indicates that the queuing system is stable if and only if the rate of left drift is larger than right drift rate (see Neuts [52]). That is

\[ \Pi B_0 e < \Pi B_2 e. \]

This gives the stability condition as

**Lemma 6.2.1.** The system under study is stable if and only if

\[ \lambda < (H \otimes I_m)e \quad (6.1) \]
6.2.2 Steady-state probability vector

Assuming that equation (6.1) is satisfied, we briefly outline the computation of the steady-state probability of the system. Let $\mathbf{y}$ denote the steady-state probability vector of the generator $Q$. Then

$$\mathbf{y}Q = 0, \quad \mathbf{ye} = 1. \quad (6.2)$$

Assuming that the stability condition (6.1) holds and partitioning $\mathbf{y}$ as

$$\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots)$$

with

$$\mathbf{y}_i = \begin{cases} y_0(a), & 1 \leq a \leq m, i = 0 \\ y_1(j, 1, j, a) \cup y_2(j, 2, \ell, a) \cup y_3(j, 3, j, a), & 1 \leq j, \ell \leq n; \ell \neq j, \\ & 1 \leq a \leq m, i \geq 1 \end{cases}$$

we obtain

$$\mathbf{y}_n = \mathbf{y}_1 R^{n-1}, n \geq 2$$

where $R$ is the minimal non negative solution to the matrix quadratic equation

$$R^2 B_2 + RB_1 + B_0 = 0.$$

The two boundary equations involving $\mathbf{y}_0$ are

$$\mathbf{y}_0 B_{00} + \mathbf{y}_1 B_{10} = 0,$$

$$\mathbf{y}_0 B_{01} + \mathbf{y}_1 [B_1 + RB_2] = 0.$$ 

These together with the normalizing condition in (6.2) gives

$$\mathbf{y}_1 = \mathbf{y}_0 V \text{ where } V = -B_{01}[B_1 + RB_2]^{-1}$$

$$\mathbf{y}_0 [1 + V(I - R)^{-1}e] = 1.$$
### 6.2.3 Expected service time of a customer

Let $i$ be the required/correct service of tagged customer. Consider the Markov chain $\{(N_2(t), N_3(t), N_4(t))/t \geq 0\}$ with state space $\{(i, j, k)/j = 1, 3; k = i\} \cup \{(i, 2, k)/1 \leq k \neq i \leq n\} \cup \{\Delta_1\} \cup \{\Delta_2\}$, where $\{\Delta_1\}$ denotes the absorbing state which is completion of service from the incorrect phases of service before the threshold clock is expired and $\{\Delta_2\}$ the absorbing state which represents the realization of the random threshold clock (that is, expulsion from service). The infinitesimal generator of this CTMC is

$$
W_i = \begin{pmatrix}
T_i & T_{\mu_i}^0 & T_{\gamma}^0 \\
0 & 0 & 0 \\
0 & 0 & -\mu_i
\end{pmatrix}
$$

where

$$
T_i = \begin{pmatrix}
-\mu_i & 0 & 0 \\
0 & S_i(i) & S_i^0 \\
0 & 0 & -\mu_i
\end{pmatrix}, \quad T_{\mu_i}^0 = \begin{pmatrix}
\mu_i \\
0 \\
\mu_i
\end{pmatrix}, \quad T_{\gamma}^0 = \begin{pmatrix}
0 \\
\gamma e \\
0
\end{pmatrix}.
$$

The service time of a customer is the time until absorption of the Markov chain. The distribution of $W_i$ is phase type with initial probability vector $\alpha_i = \left(\theta, \hat{\theta}\beta_i, 0\right)$ of order $n + 1$. The expected time a tagged customer spends in service is $E_{W_i} = -\alpha_i T_i^{-1} e$. Therefore the service time of an arbitrarily chosen customer is

$$
E_{st} = \sum_{i=1}^{n} p_i E_{W_i}.
$$

### 6.2.4 Performance measures

Now we look at a few of the system performance measures. Let a customer enters in to incorrect service with initial probability vector $(\psi_1, \psi_2, ..., \psi_n)$, $\psi_i$ being $p_i\beta(i)$ for $1 \leq i \leq n$. Let $l = (n + 1)d$.

1. Probability that the system is idle, $P_{idle} = y_0 e$. 

2. Probability that the server is busy in direct correct mode,
   \[ Y_1 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} \sum_{a=1}^{m} y_i(j,1,j,a). \]
3. Probability that the server is serving in the incorrect mode,
   \[ Y_2 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} \sum_{\ell, \ell \neq j}^{n} \sum_{a=1}^{m} y_i(j,2,\ell,a). \]
4. Probability that the server is busy in correct, service of which started in incorrect mode,
   \[ Y_3 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} \sum_{a=1}^{m} y_i(j,3,j,a). \]
5. Expected number of customers in the system,
   \[ \mu_{NS} = \sum_{i=1}^{\infty} i y_i e. \]
6. Expected number of customers in the queue,
   \[ \mu_{NQ} = \sum_{i=2}^{\infty} (i-1) y_i e. \]
7. Probability of customers leaving with correct service starting in incorrect service mode,
   \[ P_{cs} = \hat{\theta} \sum_{i=1}^{n} \psi_i (-S_i)^{-1} S_0^i. \]
8. Rate at which customers leave with correct service initially starting in incorrect service mode,
   \[ R_{cs} = \lambda P_{cs}. \]
9. Probability that a customer is lost (leaving the system without getting correct service),
   \[ P_{loss} = \hat{\theta} \sum_{i=1}^{n} \psi_i (-S_i)^{-1} \gamma e. \]
10. Rate of loss of customers due to incorrect service,
    \[ R_{loss} = \lambda P_{loss}. \]
11. Rate of customers leaving successfully after being selected in correct service,
    \[ P_{lc} = \lambda \theta. \]

### 6.3 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.
We fix parameters $n = 4, (p_1, p_2, p_3, p_4) = (0.1, 0.2, 0.3, 0.4), \mu_1 = 8, \mu_2 = 9, \mu_3 = 8, \mu_4 = 9,$

$$S^{(1)} = \begin{bmatrix} * & 6 & 8 \\ 7 & * & 6 \\ 5 & 9 & * \end{bmatrix}, S^0_1 = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}, \beta^{(1)} = \begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix},$$

$$S^{(2)} = \begin{bmatrix} * & 7 & 6 \\ 8 & * & 6 \\ 5 & 9 & * \end{bmatrix}, S^0_2 = \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix}, \beta^{(2)} = \begin{bmatrix} 0.2 & 0.4 & 0.4 \end{bmatrix},$$

$$S^{(3)} = \begin{bmatrix} * & 6 & 6 \\ 5 & * & 8 \\ 5 & 5 & * \end{bmatrix}, S^0_3 = \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix}, \beta^{(3)} = \begin{bmatrix} 0.1 & 0.5 & 0.4 \end{bmatrix},$$

$$S^{(4)} = \begin{bmatrix} * & 6 & 7 \\ 5 & * & 6 \\ 8 & 7 & * \end{bmatrix}, S^0_4 = \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix}, \beta^{(4)} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 4.96099 & 0 & 0.05011 \\ 11.2875 & 0 & 1117.4625 \end{bmatrix}.$$  

**Effect of $\theta$**

The entries in Table 6.1 are on expected lines: $P_{idle}$ increases with increasing value of $\theta$ - this means that customers, when selected in incorrect mode of service, spent a long time in the system before departure; the value of $Y_1$ steadily increases with $\theta$ since, for example all customers are selected for correct service at the beginning stage itself when $\theta = 1$; values of $Y_2$ and $Y_3$ decrease with increase in value of $\theta$, as expected and when $\theta = 1$, both turn out to be zero; in $P_{cs}$ column all entries against corresponding values of $\theta$ decrease and reach zero when $\theta = 1$. Loss
probability of customers, when admitted first to undesirable phases of service who leave without going to desired phase of service, decrease with increasing value of $\theta$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$\theta$ & $P_{idle}$ & $Y_1$ & $Y_2$ & $Y_3$ & $P_{cs}$ & $P_{loss}$ & $E_{st}$ \\
\hline
0.5 & 0.0647 & 0.2917 & 0.3625 & 0.2811 & 0.4819 & 0.0181 & 0.1871 \\
0.6 & 0.1351 & 0.3500 & 0.2900 & 0.2249 & 0.3855 & 0.0145 & 0.1730 \\
0.7 & 0.2055 & 0.4083 & 0.2175 & 0.1687 & 0.2891 & 0.0109 & 0.1589 \\
0.8 & 0.2759 & 0.4667 & 0.1450 & 0.1124 & 0.1928 & 0.0072 & 0.1448 \\
0.9 & 0.3463 & 0.5250 & 0.0725 & 0.0562 & 0.0964 & 0.0036 & 0.1307 \\
1 & 0.4167 & 0.5833 & 0 & 0 & 0 & 0 & 0.1167 \\
\hline
\end{tabular}
\caption{Effect of $\theta$ for $\gamma = 0.25$.}
\end{table}

**Effect of $\gamma$**

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$\gamma$ & $P_{idle}$ & $Y_1$ & $Y_2$ & $Y_3$ & $P_{cs}$ & $P_{loss}$ & $R_{lc}$ & $E_{st}$ \\
\hline
0.1 & 0.1969 & 0.4083 & 0.2223 & 0.1724 & 0.9852 & 0.0148 & 3.5 & 0.1606 \\
0.2 & 0.2027 & 0.4083 & 0.2191 & 0.1699 & 0.9708 & 0.0292 & 3.5 & 0.1595 \\
0.3 & 0.2083 & 0.4083 & 0.2159 & 0.1675 & 0.9568 & 0.0432 & 3.5 & 0.1583 \\
0.4 & 0.2137 & 0.4083 & 0.2128 & 0.1651 & 0.9432 & 0.0568 & 3.5 & 0.1573 \\
0.5 & 0.2190 & 0.4083 & 0.2098 & 0.1628 & 0.9301 & 0.0699 & 3.5 & 0.1562 \\
0.6 & 0.2242 & 0.4083 & 0.2069 & 0.1605 & 0.9172 & 0.0828 & 3.5 & 0.1552 \\
\hline
\end{tabular}
\caption{Effect of $\gamma$ for $\theta = 0.7$.}
\end{table}

The output in Table 6.2 also are on expected lines. Note that $P_{loss}$ increases with increasing value of $\gamma$, since clock realized faster for higher value of $\gamma$. The column corresponding to $Y_1$ has all entries with same value; this is so since the clock realization time does not affect the probability of getting into correct conditions.
service. Columns corresponding to $Y_2$ and $Y_3$ should have values decreasing with $\gamma$ increasing since faster clock realization leads to moving out of incorrect service states faster. Further $P_{cs}$ decrease with increase in value of $\gamma$. This is due again to the fact that clock realizes faster for larger values of $\gamma$, resulting in customers at undesirable phases of service leave the system (due to clock realization).

**Effect of arrival process**

For the arrival process, we consider the following five sets of values for $D_0$ and $D_1$ as follows.

1. **Exponential (EXP A):**

   \[
   D_0 = \begin{bmatrix} -5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 5 \end{bmatrix}
   \]

2. **Erlang (ERLA):**

   \[
   D_0 = \begin{bmatrix} -10 & 10 \\ 0 & -10 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix}
   \]

3. **Hyper-exponential (HEXA):**

   \[
   D_0 = \begin{bmatrix} -9.5 & 0 \\ 0 & -0.95 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 8.55 & 0.95 \\ 0.855 & 0.095 \end{bmatrix}
   \]

4. **MAP with negative correlation (MNCA):**

   \[
   D_0 = \begin{bmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.05011 & 4.96099 \\ 1117.4625 & 0 & 11.2875 \end{bmatrix}
   \]

5. **MAP with positive correlation (MPCA):**

   \[
   D_0 = \begin{bmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 4.96099 & 0 & 0.05011 \\ 11.2875 & 1117.4625 \end{bmatrix}
   \]
A MAP/PH/1 queue with uncertainty in selection of type of service

The above MAP processes will be normalized so as to have a specific arrival rate. However, these are qualitatively different in that they have different variance and correlation structure. The first three arrival processes, namely, EXPA, ERLA and HEXA have zero correlation for two successive inter-arrival times. The arrival processes labeled MNCA and MPCA, respectively, have negative and positive correlation for two successive inter-arrival times with values -0.48891 and 0.48891. The standard deviation of the inter-arrival times of these five arrival processes are, respectively, 0.2, 0.14142, 0.44894, 0.2819 and 0.2819.

The main comparison in Tables 6.3 and 6.4 is between values of $\mu_{NS}$ in MNCA and MPCA. Both decrease with increase in value of $\gamma$ ($\theta$) in both tables. However, MNCA has much smaller values compared to their MPCA counter parts. This indicates that positive correlation in the arrival process results in accumulation of large number of customers in the system.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>EXPA</th>
<th>ERLA</th>
<th>HEXA</th>
<th>MNCA</th>
<th>MPCA</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.0165</td>
<td>3.0730</td>
<td>10.5391</td>
<td>4.2115</td>
<td>203.9659</td>
</tr>
<tr>
<td>0.2</td>
<td>3.8699</td>
<td>2.9622</td>
<td>10.1056</td>
<td>4.0638</td>
<td>196.7258</td>
</tr>
<tr>
<td>0.3</td>
<td>3.7355</td>
<td>2.8609</td>
<td>9.7083</td>
<td>3.9285</td>
<td>190.0795</td>
</tr>
<tr>
<td>0.4</td>
<td>3.612</td>
<td>2.7677</td>
<td>9.3429</td>
<td>3.8041</td>
<td>184.2689</td>
</tr>
<tr>
<td>0.5</td>
<td>3.498</td>
<td>2.6819</td>
<td>9.0058</td>
<td>3.6892</td>
<td>178.2982</td>
</tr>
</tbody>
</table>

Table 6.3: Effect of $\gamma$ for $\theta = 0.7$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>EXPA</th>
<th>ERLA</th>
<th>HEXA</th>
<th>MNCA</th>
<th>MPCA</th>
</tr>
</thead>
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Table 6.4: Effect of $\theta$ for $\gamma = 0.25$
Concluding remarks and suggestions for future study:

In this thesis we discussed priority queueing models with self generation of lower priorities through interruption or feedback. A multi server priority model in the context of crowdsourcing was analyzed. Also discussed are queueing systems where uncertainty prevails in the selection of service.

Chapter 2 dealt with a highly dependent priority queueing system where low priority customers join the queue from immediately preceding waiting lines due to interruption of service by self. We assumed all underlying distributions to be exponential. Analytical expressions for system state probabilities were computed. The second chapter discussed an analogous situation but customers joined the low priority queue only after completing their service from high priority line. A multi server priority queueing model with two types of customers was discussed in chapter 4. The main advantage with the problem we analyzed in this chapter, in comparison with that of Chakravarthy and Dudin [11] is that the loss of high priority customers is reduced due to preemption. This results in a larger number of low priority customers being served by high priority customers. However, pre-emption of a low priority, sometimes even more than once, may lead to its longer waiting time in the system. Nevertheless if suitable incentive is provided to the high priority customer who serve a low priority customer on leaving the system, the probability to offer service may become close to 1, if not equal to 1. The thesis then focused some diagnostic problems where uncertainty in the selection of service type plays a prominent role. In chapter 5 we analyzed a situation where service starts without knowing whether it is going to be inappropriate for the customer, but service is compulsorily needed for customer arriving at the service point. We assumed the case of two types of services of which one is correct and services are offered in phases. In the last chapter we examined a queueing model offering $n$ distinct services, but for any customer one among the $n$ services was required and the remaining $n-1$ were damaging (undesirable/ inessential) and
both of these cases were analyzed for a single server case.

In a future work we propose to extend the models in chapters 2 and 3 to the case of correlated arrivals. Crowdsourcing model is to be analyzed in the context of queueing-inventory scenario. In the diagnostic problems further analysis is needed when required service constitutes more than one correct service. The advantage in using multiple service channels to improve the performance of the system is to be explored. Also, analysis of the case of arbitrarily distributed service process is under progress.