3.1 Introduction

Algebraic approach to fuzzy automata theory mostly depends on the finite monoid theory because of the one-one correspondence between a fuzzy finite state automaton and its transition monoid. Eilenberg type variety theorem for fuzzy languages says that there is a one-one correspondence between variety of finite monoids and the variety of regular fuzzy languages. We know that the collection of finite inverse monoids does not form a variety since subalgebra of an inverse monoid need not be an in-

---

Some results of this chapter are included in the following paper.
verse monoid. But they generate a variety called InV. InV consists of all finite monoids with commuting idempotents[2]. In this chapter we define an inverse fuzzy automaton such that its transition monoid is an inverse monoid and study some of its algebraic properties. We define an inverse fuzzy language, give a characterization for inverse fuzzy languages and prove some results. Also we prove some properties of inverse fuzzy languages based on the fact that the syntactic monoid is an inverse monoid.

3.2 Preliminaries

Definition 3.2.1. A semigroup $S$ is called regular if for every element $a$ in $S$ there exist $a$ $b$ in $S$ such that $a = aba$. A semigroup $S$ is said to be an inverse semigroup if for every $a$ in $S$ there exists a unique $b$ in $S$ such that $aba = a$ and $bab = b$. We call $b$ the inverse of $a$ and denote by $a^{-1}$. If $S$ has an identity then $S$ is said to be an inverse monoid.

For any element $a$ of an inverse monoid, $aa^{-1}$ is an idempotent and idempotents of an inverse monoid commute. The collection of all finite inverse monoids generate a variety InV which is the collection of all semigroups with commuting idempotents. This is the smallest variety containing finite inverse monoids [2]. An analogues to Cayley’s theorem for groups, Preston and Wagner proved that an inverse monoid $S$ is isomorphic to a subinverse monoid of the monoid of all one-one partial transformations on $S$. A regular semigroup can be characterized by the property that for every $x$ in $S$ the $\mathcal{L}$-class ($\mathcal{R}$-class) containing $x$ contains an idempotent and inverse semigroups can be characterized by the properties, (a) $S$
3.3. Regular and inverse fuzzy automata

Let $X$ be a nonempty finite set. Let $X^*$ be the free monoid generated by $X$. Then $X^*$ is regular if for every $x \in X^*$ there exist a $y \in X^*$ such that $x = xyx$ and $X^*$ is inverse if $\forall x \in X^*$, there exist a unique $y \in X^*$ such that $x = xyx$, $y = yxy$.

**Definition 3.3.1.** Let $M = (Q, X, \mu)$ be a fuzzy automaton. $M$ is said to be regular if for every $x \in X^*$ there exist a $y \in X^*$ such that $\mu(p, x, q) = \mu(p, xyx, q)$ for all $p, q \in Q$. $M = (Q, X, \mu)$ is said to be an inverse fuzzy automaton if $\forall x \in X^*$, there exist a unique $y \in X^*$ such that $\mu(q, xyx, p) = \mu(q, x, p)$, $\mu(q, yxy, p) = \mu(q, y, p)$ $\forall p, q \in Q$.

In the case of a deterministic inverse fuzzy automaton this can be re-defined as $\forall x \in X^*$, there exist a unique $y$ such that $\mu(q, x, p) = \mu(p, y, q)$ and $\mu(p, x, q) = \mu(r, x, q) \implies p = r \forall p, q, r \in Q$. A deterministic fuzzy automaton can be represented by the transition matrices with each row contains atmost one nonzero entry (partial fuzzy transformations) and a deterministic inverse fuzzy automaton can be represented by transition matrices with each row and column contains atmost one nonzero entry (one-one partial fuzzy transformations). For an inverse fuzzy automaton we take $X^*$ to assure the existence of such a $y$. ie, $\forall x \in X^*$, $\mu(q, xx^{-1}x, p) = \mu(q, x, p)$ and $\mu(q, x^{-1}xx^{-1}, p) = \mu(q, x^{-1}, p) \forall p, q \in Q$.

**Definition 3.3.2.** A fuzzy language $\lambda$ on an alphabet $X$ is said to
be an inverse fuzzy language if the minimal fuzzy automaton recognizing that language is an inverse fuzzy automaton.

**Example 3.3.1.** Let $M=(Q, \bar{X}, \mu, i, \tau)$, where $Q = \{q_0, q_1, q_2\}$, $\bar{X} = \{a, b\}$, $i = [1 \ 0 \ 0]$, $\tau = [0 \ 0 \ 1]^T$ and $\mu : Q \times X \times Q \rightarrow [0, 1]$ as defined below

$$
\mu(q_0, a, q_1) = 0.7, \mu(q_1, a, q_2) = 0.4, \mu(q_2, a, q_0) = 0.3, \mu(q_1, b, q_0) = 0.8, \\
\mu(q_0, b, q_2) = 0.6, \mu(q_2, b, q_1) = 0.5 \text{ and } = 0 \text{ for all other elements of } Q \times \bar{X} \times Q.
$$

This is a deterministic regular fuzzy automaton for which $T_{aba} = T_a$. But this is not an inverse fuzzy automaton since $b$ is not unique and $T_{bab} \neq T_b$.

**Example 3.3.2.** Let $M=(Q, \bar{X}, \mu, i, \tau)$, where $Q = \{q_0, q_1, q_2\}$, $\bar{X} = \{a, b\}$, $i = [1 \ 0 \ 0]$, $\tau = [0 \ 0 \ 1]^T$ and $\mu : Q \times \bar{X} \times Q \rightarrow [0, 1]$ as defined below

$$
\mu(q_0, a, q_1) = 0.7, \mu(q_1, a, q_2) = 0.4, \mu(q_2, a, q_0) = 0.3, \mu(q_1, b, q_0) = 0.7, \\
\mu(q_0, b, q_2) = 0.3, \mu(q_2, b, q_1) = 0.4 \text{ and } = 0 \text{ for all other elements of } Q \times X \times Q.
$$
3.3. Regular and inverse fuzzy automata

Then \( T_a = \begin{bmatrix} 0 & 0.7 & 0 \\ 0 & 0 & 0.4 \\ 0.3 & 0 & 0 \end{bmatrix} \) \( T_b = \begin{bmatrix} 0 & 0 & 0.3 \\ 0.7 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \)

Then, the transition semigroup \( T(M) \) is the semigroup generated by \( \{T_a, T_b\} \) in which \( T_{aba} = T_a, T_{bab} = T_b \).

Thus

\[
T(M) = \{T_a, T_{a^2}, T_{a^3}, T_{a^4}, T_{a^5}, T_b, T_{b^2}, T_{ab}, T_{ba}, T_{ab^2}, T_{b^2a}, T_{ba^2}, T_{a^2b^2}, T_{b^2a^2}, T_{ab^2a}\}.
\]

Here \( T_{a^3}, T_{ab}, T_{ba}, T_{b^2a^2}, T_{a^2b^2}, T_{ab^2a} \) are the idempotents.

\[L_{T_a} = \{T_a, T_{ba}\} \]
\[L_{T_b} = \{T_b, T_{ab}\} \]
\[L_{T_{a^2}} = \{T_{a^2}, T_{ba^2}, T_{b^2a^2}\} \]
\[L_{T_{b^2}} = \{T_{b^2}, T_{ab^2}, T_{a^2b^2}\} \]
\[L_{T_{a^2b}} = \{T_{a^2b}, T_{b^2a}, T_{ab^2a}\} \]
\[L_{T_{a^3}} = \{T_{a^3}, T_{a^4}, T_{a^5}\} \]

Since every \( L \) class contains a unique idempotent, \( T(M) \) is an inverse semigroup.

The fuzzy language accepted by this fuzzy automaton is

\[
\lambda(x) = \begin{cases} 
0.4 & \text{when } x = ((aabb)^* + (ab)^*)aa \\
0.3 & \text{when } x = b((ab)^* + (ba)^*) + (bbb)^*(aa + b + bba) \\
0 & \text{for all other } x \in X^* 
\end{cases}
\]

which is an inverse fuzzy language.

**Example 3.3.3.** Let \( Q = \{q_0, q_1, q_2\}, X = \{a, b\} \). Take \( \tilde{X}^* \) as the free inverse monoid generated by \( X \). Consider a deterministic fuzzy automaton \( M = (Q, \tilde{X}, \mu) \) over \( \tilde{X} = X \cup X^{-1} \) where \( \mu \) is a fuzzy subset of \( Q \times \tilde{X} \times Q \) with finite image \( C \) such that for every \( p \in Q \), there exist at most one \( q \in Q \) such that \( \mu(q, a, p) > 0 \) and \( \mu(q, a, p) = \mu(p, a^{-1}, q) \) \( \forall \) \( p, q \in Q, a \in X \). Then \( M \) is a deterministic fuzzy automaton which is inverse. Here \( \tilde{X}^* \) acts on \( Q \) as one-one partial fuzzy transformations. The transition monoid
is a subinverse monoid of the inverse monoid of all one-one partial fuzzy transformations on $Q$.

**Theorem 3.3.1.** A fuzzy automaton $A = (Q, \tilde{X}, \mu, i, \tau)$ is inverse (regular) if and only if $\tilde{X}^*/\theta_A$ is an inverse (regular) monoid.

**Proof.** Suppose $A$ is an inverse fuzzy automaton

ie, for each $x \in \tilde{X}^*$ there exist a unique $x^{-1} \in \tilde{X}^*$ such that $\forall p, q \in Q, \mu(q, xx^{-1}x, p) = \mu(q, x, p)$ and $\mu(q, x^{-1}xx^{-1}, p) = \mu(q, x^{-1}, p)$

$\iff xx^{-1}x \theta_A x$ and $x^{-1}xx^{-1} \theta_A x^{-1}$

$\iff [xx^{-1}x] = [x]$ and $[x^{-1}xx^{-1}] = [x^{-1}]$.

Then, $[x][x^{-1}][x] = [x]$ and $[x^{-1}][x][x^{-1}] = [x^{-1}]$

$\iff \tilde{X}^*/\theta_A$ is an inverse monoid.

As a particular case it is true that $A$ is regular if and only if $\tilde{X}^*/\theta_A$ is regular.

$\square$

### 3.4 Construction of regular and inverse fuzzy automata

Since every nondeterministic fuzzy automaton can be converted into a deterministic fuzzy automaton we give the construction of a deterministic inverse fuzzy automaton. There is a one to one correspondence between finite inverse (regular) monoids and inverse (regular) fuzzy automata on the set of generators. To construct a deterministic inverse fuzzy automaton
3.4. Construction of regular and inverse fuzzy automata

on \( n \) states \( Q = \{q_1, q_2, \ldots, q_n\} \), take \( C = \{c_1, c_2, \ldots, c_k\}, \quad k \leq n, c_i \in [0, 1] \). Consider the collection of all matrices with entries in \( C \) and such that there exists atmost one non-zero entry in each row and column. This collection represent the set of all one-one partial fuzzy transformations on \( Q \) with image in \( C \), which is a monoid under the \( \max - \min \) operation, denoted by \( \text{FI}^C_Q \). \( \text{FI}^C_Q \) is finite since \( Q \) and \( C \) are finite. For every \( A \in \text{FI}^C_Q \), there exists a unique inverse \( B \in \text{FI}^C_Q \) such that \( ABA = A \) and \( BAB = B \). Here \( B \) will be the transpose of \( A \). To construct an inverse fuzzy automaton on a finite alphabet \( m \), take \( m \) matrices from \( \text{FI}^C_Q \) such that transpose of each matrices is included in the collection. Construct an automaton with these matrices as the transition matrices of the \( m \) alphabets. The automaton will be an inverse fuzzy automaton with the transition monoid as the monoid generated by the chosen fuzzy matrices.

For a deterministic regular fuzzy automaton, we take a fuzzy matrix \( A = [a_{ij}] \in \text{FI}^C_Q \) and \( B = [b_{ij}] \) is another fuzzy matrix in \( \text{FI}^C_Q \) such that \( b_{ij} \geq a_{ji} \) for all \( a_{ji} \neq 0 \) and \( = 0 \) for \( a_{ji} = 0 \). Then \( ABA = A \) but \( B \) is not unique and \( BAB \) need not be equal to \( B \).

**Example 3.4.1.** Let \( c_0, c_1, c_2, c_4 \in [0, 1] \) \( Q = \{q_0, q_1, q_2\} \), \( X = \{a, b\} \)

\[
T_a = \begin{pmatrix} 0 & c_0 & 0 \\
0 & 0 & 0 \\
0 & 0 & c_1 \end{pmatrix}, \quad T_{a^{-1}} = \begin{pmatrix} 0 & 0 & 0 \\
0 & c_0 & 0 \\
0 & 0 & c_1 \end{pmatrix}
\]

\[
T_b = \begin{pmatrix} c_2 & 0 & 0 \\
0 & 0 & c_3 \\
0 & c_4 & 0 \end{pmatrix}, \quad T_{b^{-1}} = \begin{pmatrix} c_2 & 0 & 0 \\
0 & 0 & c_4 \\
0 & c_3 & 0 \end{pmatrix}
\]

Then \( M = (Q, \tilde{X}, \{T_a, a \in \tilde{X}\}) \) is an inverse fuzzy automaton.
3.5 Inverse fuzzy languages

We have proved that the transition monoid of an inverse fuzzy automaton is an inverse monoid. If a fuzzy language is recognized by an inverse fuzzy automaton, the corresponding transition monoid should recognize that fuzzy language. So if $\lambda$ is an inverse fuzzy language on $\tilde{X}$, there exist an inverse monoid $I$ and a fuzzy subset $\delta$ of $I$ and a homomorphism $\phi$ from $\tilde{X}^*$ to $I$ such that $\phi^{-1}(\delta) = \lambda$. i.e, $\phi^{-1}(\delta)(u) = \lambda(u) \forall u \in \tilde{X}^*$.

**Theorem 3.5.1.** (Characterization of an inverse fuzzy language) A fuzzy language $\lambda$ on $\tilde{X}$ is inverse if and only if for every $x \in \tilde{X}^*$
$\lambda(xx^{-1}xv) = \lambda(xv)$ and $\lambda(xx^{-1}x^{-1}v) = \lambda(x^{-1}v)$ for every $u, v \in \tilde{X}^*$.

**Proof.** Let $\lambda$ is a regular fuzzy language on $\tilde{X}$. The transition monoid of the minimal automaton $M(\lambda)$ is the syntactic monoid of $\lambda$. Let $P_\lambda$ be the main congruence on $\tilde{X}^*$ defined by $xP_\lambda y$ if and only if $\lambda(uxv) = \lambda(uyv)$ for all $u, v \in \tilde{X}^*$. The transition monoid of the quotient fuzzy automaton is isomorphic to the syntactic monoid of the fuzzy language $\lambda$. Thus $\tilde{X}^*/P_\lambda$ is isomorphic to $\tilde{X}^*/\theta_M$. Suppose $\lambda$ is an inverse fuzzy language. Then the minimal fuzzy automaton $M(\lambda)$ recognizing $\lambda$ is an inverse fuzzy automaton. i.e, $\tilde{X}^*/\theta_M$ is an inverse monoid. Then for each $x \in \tilde{X}^*$ there exist a unique $x^{-1} \in \tilde{X}^*$ such that $\forall p, q \in Q$,
\[
\mu(q, xx^{-1}x, p) = \mu(q, x, p) \quad \text{and} \quad \mu(q, x^{-1}xx^{-1}, p) = \mu(q, x^{-1}, p)
\]
\[
\implies [xx^{-1}x]_{\theta_M} = [x]_{\theta_M} \quad \text{and} \quad [x^{-1}xx^{-1}]_{\theta_M} = [x^{-1}]_{\theta_M}
\]
\[
\implies [xx^{-1}x]_{P_\lambda} = [x]_{P_\lambda} \quad \text{and} \quad [x^{-1}xx^{-1}]_{P_\lambda} = [x^{-1}]_{P_\lambda}
\]
\[
\implies xx^{-1}xP_\lambda x \quad \text{and} \quad x^{-1}xx^{-1}P_\lambda x^{-1}
\]
\[
\implies \lambda(uxx^{-1}xv) = \lambda(uxv) \quad \text{and} \quad \lambda(u^{-1}xx^{-1}v) = \lambda(u^{-1}v) \quad \forall u, v \in X^*.
\]
Conversely, suppose for every \( x \in \tilde{X}^* \), \( \lambda(uxx^{-1}xv) = \lambda(uxv) \) and \( \lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v) \) for every \( u, v \in \tilde{X}^* \).

\[
xx^{-1}xP_\lambda x \text{ and } x^{-1}xx^{-1}P_\lambda x^{-1}
\]

\[
[xx^{-1}x]P_\lambda = [x]P_\lambda \text{ and } [x^{-1}xx^{-1}]P_\lambda = [x^{-1}]P_\lambda
\]

\[
[xx^{-1}]_{\theta M} = [x]_{\theta M} \text{ and } [x^{-1}xx^{-1}]_{\theta M} = [x^{-1}]_{\theta M}
\]

\[
\tilde{X}^*/\theta M \text{ is an inverse monoid. i.e, the minimal fuzzy automaton recognizing } \lambda \text{ is an inverse automaton and thus } \lambda \text{ is an inverse fuzzy language.}
\]

\textbf{Lemma 3.5.1.} The set of all inverse fuzzy languages on an alphabet \( \tilde{X} \) is closed under intersection.

\textbf{Proof.} Let \( \lambda_1 \) and \( \lambda_2 \) be two inverse fuzzy languages in \( \tilde{X}^* \).

Then there exist two inverse fuzzy automata \( M_1 = (Q_1, \tilde{X}, \mu_1, i_1, \tau_1) \) and \( M_2 = (Q_2, \tilde{X}, \mu_2, i_2, \tau_2) \) recognizing \( \lambda_1 \) and \( \lambda_2 \) respectively. Then the restricted direct product \( M_1 \times M_2 \) is an inverse fuzzy automaton since

\[
\mu_1 \times \mu_2((q_1, q_2), x, (p_1, p_2)) = \mu_1(q_1, x, p_1) \wedge \mu_2(q_2, x, p_2) \forall x \in \tilde{X}^* \text{ and } M_1 \text{ and } M_2 \text{ are inverse fuzzy automata. The language recognized by } M_1 \times M_2 \text{ is } \lambda_1 \wedge \lambda_2 \text{ [14].}
\]

So \( \lambda_1 \wedge \lambda_2 \) is an inverse fuzzy language.

\textbf{Lemma 3.5.2.} Let \( M_1 = (Q_1, \tilde{X}, \mu_1, i_1, \tau_1) \) and \( M_2 = (Q_2, \tilde{X}, \mu_2, i_2, \tau_2) \) be two inverse fuzzy automata with \( Q_1 \cap Q_2 = \phi \) and recognizing the inverse fuzzy languages \( \lambda_1 \) and \( \lambda_2 \) respectively. Then their join \( M_1 \vee M_2 \) is an inverse fuzzy automaton recognizing \( \lambda_1 \vee \lambda_2 \).

\textbf{Proof.} We have
\[ \lambda_1(x) = \bigvee_{p,q \in Q_1} i_1(p) \land \mu_1(p, x, q) \land \tau_1(q) \] and
\[ \lambda_2(x) = \bigvee_{p,q \in Q_2} i_2(p) \land \mu_2(p, x, q) \land \tau_2(q). \]

Now the fuzzy language recognized by \( M_1 \lor M_2 \) is
\[ \lambda(x) = \bigvee_{p,q \in Q_1 \cup Q_2} (i_1 \lor i_2)(p) \land (\mu_1 \lor \mu_2)(p, x, q) \land (\tau_1 \lor \tau_2)(q) \]
\[ = \bigvee_{p,q \in Q_1} i_1(p) \land \mu_1(p, x, q) \land \tau_1(q) \lor \bigvee_{p,q \in Q_2} i_2(p) \land \mu_2(p, x, q) \land \tau_2(q) \]
\[ = \lambda_1(x) \lor \lambda_2(x) \]
\[ = (\lambda_1 \lor \lambda_2)(x) \text{ for all } x \in \tilde{X}^*. \]

Also we have \( \forall x \in \tilde{X}^*, \mu_1 \lor \mu_2(p, x, q) = \begin{cases} 
\mu_1(p, x, q) & \text{if } p, q \in Q_1 \\
\mu_2(p, x, q) & \text{if } p, q \in Q_2 \\
0 & \text{otherwise.} 
\end{cases} \)

So if \( M_1 \) and \( M_2 \) are two inverse fuzzy automata then their join \( M_1 \lor M_2 \) is an inverse fuzzy automaton.

\[ \square \]

**Theorem 3.5.2.** The class of all inverse fuzzy languages in \( \tilde{X}^* \) is closed under finite boolean operations.

**Proof.** Let \( \lambda \) be an inverse fuzzy language.

Then by the characterization of inverse fuzzy language for every \( x \in \tilde{X}^* \) there exist \( x^{-1} \in \tilde{X}^* \) such that \( \lambda(uxx^{-1}xv) = \lambda(uxv) \) and
\[ \lambda(u.x^{-1}xv) = \lambda(u.x^{-1}v) \forall u, v \in \tilde{X}^*. \]
3.6. Homomorphic image of inverse fuzzy automata

Then \( \lambda^c(uxx^{-1}xv) = 1 - \lambda(uxx^{-1}xv) = 1 - \lambda(uxv) = \lambda^c(uxv) \) and
\( \lambda^c(ux^{-1}xx^{-1}v) = 1 - \lambda(ux^{-1}xx^{-1}v) = 1 - \lambda(uvx) = \lambda^c(uvx) \).

Thus \( \lambda^c \) is an inverse fuzzy language.

Closure properties of union and intersection of inverse fuzzy languages follows by Lemma 3.5.1 and Lemma 3.5.2.

\[ \square \]

3.6 Homomorphic image of inverse fuzzy automata

**Definition 3.6.1.** Let \( M_1 = (Q_1, X_1, \mu_1, i_1, \tau_1) \) and \( M_2 = (Q_2, X_2, \mu_2, i_2, \tau_2) \) be two fuzzy automata. A pair \((\alpha, \beta)\) of mappings \( \alpha : Q_1 \rightarrow Q_2 \) and \( \beta : X_1 \rightarrow X_2 \) is called a homomorphism written as \((\alpha, \beta) : M_1 \rightarrow M_2\) if \( \mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p)) \) \( \forall p, q \in Q_1 \) and \( \forall x \in X_1 \). The pair \((\alpha, \beta)\) is called a strong homomorphism if \( \mu_2(\alpha(q), \beta(x), \alpha(p)) = \bigvee \{ \mu_1(q, x, t : t \in Q_1, \alpha(t) = \alpha(p)) \} \) \( \forall q, p \in Q_1 \) and \( \forall x \in X_1 \). \( \beta \) can be extended to \( \beta^* : X_1^* \rightarrow X_2^* \) by \( \beta^*(\Lambda) = \Lambda \) and \( \beta^*(ua) = \beta^*(u)\beta^*(a) \) \( \forall u \in X_1^*, a \in X_1 \) and \( \beta^*(uv) = \beta^*(u)\beta^*(v) \) \( \forall u, v \in X^* \) [14]. If \( \alpha, \beta \) are one-one and onto then \((\alpha, \beta)\) is called an isomorphism.

**Theorem 3.6.1.** Let \( M_1, M_2 \) be two fuzzy automata. Let \((\alpha, \beta) : M_1 \rightarrow M_2\) be strong homomorphism. Then \( \alpha \) is one-one if and only if \( \mu_1(q, x, p) = \mu_2(\alpha(q), \beta^*(x), \alpha(p)) \) \( \forall q, p \in Q \) and \( x \in X_1^* \) [14].

**Theorem 3.6.2.** If \( M_1 = (Q_1, \tilde{X}_1, \mu_1) \) and \( M_2 = (Q_2, \tilde{X}_2, \mu_2) \) be two fuzzy automata. Let \((\alpha, \beta) : M_1 \rightarrow M_2\) be a strong homomorphism from
that

µ

µ

M

56 Chapter 3. Regular and Inverse Fuzzy Automata

p,q

∈

M

Since

And,

x

Let

β

Then

Then there exists

u

β

If

λ

x

Since

Thus the image of

M

Thus

(µ,q,xyx,p) = µ1(q,x,p) and µ1(q,yxy,p) = µ1(q,y,p) for every

p,q ∈ Q.

µ2(α(q),β*(x)β*(y)β*(x),α(p)) = µ2(α(q),β*(xyx),α(p))

= \bigvee \{µ1(q,xyx,t) : t ∈ Q, α(t) = α(p)\}

= \bigvee \{µ1(q,x,t) : t ∈ Q, α(t) = α(p)\}

= µ2(α(q),β*(x),α(p))

And,

µ2(α(q),β*(y)β*(x)β*(y),α(p)) = µ2(α(q),β*(yxy),α(p))

= \bigvee \{µ1(q,yxy,t) : t ∈ Q, α(t) = α(p)\}

= \bigvee \{µ1(q,y,t) : t ∈ Q, α(t) = α(p)\}

= µ2(α(q),β*(y),α(p))

Thus the image of

M

under

(µ,q) is an inverse fuzzy automata.

If \( λ \) is a fuzzy language recognized by

M

then its image \( β^*(λ) \) defined as

\[
β^*(λ)(u) = \begin{cases}
\bigvee \{λ(w) : β^*(w) = u \text{ if } β^{−1}(u) \neq φ\} & \text{for every } u ∈ \tilde{X}^*_2 \\
0 & \text{otherwise}
\end{cases}
\]

Let \( x ∈ \tilde{X}^*_2 \).

Then \( β^*(λ)(uxv) = \begin{cases}
\bigvee \{λ(w) : β^*(w) = uxv \text{ if } β^{−1}(uxv) \neq φ\} & \\
0 & \text{otherwise.}
\end{cases} \)

Then there exists \( u', x', v' ∈ \tilde{X}^*_1 \) such that \( β^*(u'x'v') = uxv \).

Since \( x' ∈ \tilde{X}^*_1 \), there exists a unique inverse \( y' ∈ \tilde{X}^*_1 \) such that

\( λ(u'x'y') = λ(u'x'y'x'v') \) and \( λ(u'y'v') = λ(u'y'x'y'v') \ ∀ u', v' ∈ \tilde{X}^*_1 \).
3.6. Homomorphic image of inverse fuzzy automata

So \( \beta^*(\lambda)(uxv) = \begin{cases} \bigvee \{ \lambda(u'x'v') : \beta^*(u'x'v') = uxv \text{ if } \beta^{*^{-1}}(uxv) \neq \phi \} \\ 0 \text{ otherwise} \end{cases} \)

\[ = \begin{cases} \bigvee \{ \lambda(u'x'y'x'v') : \beta^*(u'x'y'x'v') = uxyxv \text{ if } \beta^{*^{-1}}(uxyxv) \neq \phi \} \\ 0 \text{ otherwise} \end{cases} \]

\[ = \beta^*(\lambda)(uxyxv) \]

Similarly we can prove that \( \beta^*(\lambda)(uyxv) = \beta^*(\lambda)(uyv) \).

This says that \( \beta^*(\lambda) \) is an inverse fuzzy language.

\[ \square \]

**Theorem 3.6.3.** If \( M_1 = (Q_1, \tilde{X}_1, \mu_1) \) and \( M_2 = (Q_2, \tilde{X}_2, \mu_2) \) be two fuzzy automata. Let \( (\alpha, \beta) : M_1 \longrightarrow M_2 \) be a strong homomorphism from \( M_1 \longrightarrow M_2 \) with \( \alpha, \beta \) are one-one and onto and if \( M_2 \) is inverse. Then \( (\alpha, \beta)^{-1}(M_2) \) is also inverse.

**Proof.** Suppose \( (\alpha, \beta) \) be a strong homomorphism with \( \alpha \), being one one onto. Then \( (\alpha, \beta) : M_1 \longrightarrow M_2 \) has the property

\[ \mu_2(\alpha(q), \beta^*(x), \alpha(p)) = \mu_1(q, x, p) \ \forall x \in \tilde{X}_1^* \]

Let \( M_2 \) be an inverse fuzzy automata.

Then for every \( x \in \tilde{X}_2^* \) there exists a unique \( y \in \tilde{X}_2^* \) such that

\[ \mu_2(q, xyx, p) = \mu_2(q, x, p) \text{ and } \mu_2(q, yxy, p) = \mu_2(q, y, p) \ \forall q, p \in Q_2. \]

ie, \( \mu_1(\alpha^{-1}(q), \beta^{*^{-1}}(xyx), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), \beta^{*^{-1}}(x), \alpha^{-1}(p)) \)

and

\[ \Rightarrow \mu_1(\alpha^{-1}(q), \beta^{*^{-1}}(x)\beta^{*^{-1}}(y)\beta^{*^{-1}}(x), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), \beta^{*^{-1}}(x), \alpha^{-1}(p)) \]

and

\[ \Rightarrow (\alpha, \beta)^{-1}(M_2) \text{ is inverse.} \]
If \( \lambda \) is a fuzzy language recognized by \( M_2 \), then its inverse image \( \beta^{-1}(\lambda) \) defined as \( \beta^{-1}(\lambda)(u) = \lambda(\beta^*(u)) \forall u \in \bar{X}_1^* \).

Let \( u, x, v \in \bar{X}_1^* \). Then since \( \beta^* \) is an isomorphism, \( \beta^*(y) \) is the inverse of \( \beta^*(x) \) where \( y \) is the inverse of \( x \). Then,

\[
\beta^{-1}(\lambda)(uxv) = \lambda(\beta^*(uxv)) \\
= \lambda(\beta^*(u)\beta^*(x)\beta^*(v)) \\
= \lambda(\beta^*(u)\beta^*(x)\beta^*(y)\beta^*(x)\beta^*(v)) \\
= \lambda(\beta^*(uxyxv)) \\
= \beta^{-1}\lambda(uxyxv)
\]

And

\[
\beta^{-1}(\lambda)(uyv) = \lambda(\beta^*(uyv)) \\
= \lambda(\beta^*(u)\beta^*(y)\beta^*(v)) \\
= \lambda(\beta^*(u)\beta^*(y)\beta^*(x)\beta^*(y)\beta^*(v)) \\
= \lambda(\beta^*(uyxyv)) \\
= \beta^{-1}\lambda(uyxyv)
\]

So \( \beta^{-1}(\lambda) \) is an inverse fuzzy language.
3.7 Cartesian product of two inverse fuzzy automata

**Definition 3.7.1.** Let $M_1 = (Q_1, X_1, \mu_1), M_2 = (Q_2, X_2, \mu_2)$ be fuzzy finite state machines such that $Q_1 \cap Q_2 = \phi$ and $X_1 \cap X_2 = \phi$. Then their direct sum is defined as $M_1 \oplus M_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \mu_1 \oplus \mu_2)$ where

$$
\mu_1 \oplus \mu_2(p,a,q) = \begin{cases} 
\mu_1(p,a,q) & \text{if } p,q \in Q_1, a \in X_1 \\
\mu_2(p,a,q) & \text{if } p,q \in Q_2, a \in X_2 \\
1 & \text{if either } (p,a) \in Q_1 \times X_1, q \in Q_2 \\
0 & \text{otherwise}
\end{cases}
$$

and the cartesian product is defined as $M_1 \cdot M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \mu_1 \cdot \mu_2)$ where

$$
\mu_1 \cdot \mu_2((p_1,p_2),a,(q_1,q_2)) = \begin{cases} 
\mu_1(p_1,a,q_1) & \text{if } a \in X_1 \text{ and } p_2 = q_2 \\
\mu_2(p_2,a,q_2) & \text{if } a \in X_2 \text{ and } p_1 = q_1 \\
0 & \text{otherwise}
\end{cases}
$$

**Theorem 3.7.1.** Let $M_1 = (Q_1, \tilde{X}_1, \mu_1, i_1, \tau_1), M_2 = (Q_2, \tilde{X}_2, \mu_2, i_2, \tau_2)$ be two fuzzy automata. If $M_1$ and $M_2$ are inverse fuzzy automata then their Cartesian product $M_1 \cdot M_2$ is an inverse fuzzy automaton.

**Proof.** We have two theorems (see [14])

(1). If $M_1 = (Q_1, X_1, \mu_1), M_2 = (Q_2, X_2, \mu_2)$ be two fuzzy finite state machines such that $X_1 \cap X_2 = \phi$. Then for every $w \in (X_1 \cup X_2)^*$, there exist $u \in X_1^*, v \in X_2^*$ such that

$$(\mu_1 \cdot \mu_2)((p_1,p_2),w,(q_1,q_2)) = (\mu_1 \cdot \mu_2)((p_1,p_2),uv,(q_1,q_2)).$$
\(w^* = uv\) is called the standard form of \(w\).

(2). For every \(u \in X_1^*, v \in X_2^*,\)
\[(\mu_1, \mu_2)((p_1, p_2), wv, (q_1, q_2)) = \mu(p_1, u, q_1) \land \mu_2(p_2, v, q_2)\]
\[= (\mu_1, \mu_2)((p_1, p_2), uv, (q_1, q_2))\] for every \((p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2.\)

First suppose \(M_1\) and \(M_2\) are two inverse fuzzy automata.

Let \(w \in (X_1 \cup X_2)^*\).

We know that \((X_1 \cup X_2)^*\) is the free semigroup on \((X_1 \cup X_2) \cup (X_1 \cup X_2)^{-1}\)
in which \(w = vw^{-1}w\) and \(w^{-1} = w^{-1}vw^{-1}\) for all \(w \in X_1 \cup X_2.\)

If \(w = \Lambda\) then the proof is trivial.

Suppose \(w \neq \Lambda.\)

Case 1. Let \(w \in (X_1 \cup X_2)^*\).

By the above theorem there exist \(u \in X_1^*, v \in X_2^*\) such that
\[(\mu_1, \mu_2)((p_1, p_2), w, (q_1, q_2)) = (\mu_1, \mu_2)((p_1, p_2),uv,(q_1,q_2))\]
\[= \mu(p_1, u, q_1) \land \mu_2(p_2, v, q_2)\] for every \((p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2.\) Since \(M_1\) and \(M_2\) are inverse fuzzy automata, there exist unique symmetric inverses \(u^{-1} \in X_1^{-1}\) and \(v^{-1} \in X_2^{-1}\) such that
\[\mu_1(p_1, u, q_1) = \mu_1(p_1, uu^{-1}u, q_1)\] and \(\mu_2(p_2, v, q_2) = \mu_2(p_2, vv^{-1}v, q_2)\) for every \(p_1, q_1 \in Q_1, p_2, q_2 \in Q_2.\)

Let \(w^{-1} = v^{-1}u^{-1}.\) Then clearly \(w^{-1} \in (X_1 \cup X_2)^{-1}\) and
\[(\mu_1, \mu_2)((p_1, p_2), w, (q_1, q_2))\]
\[= \mu(p_1, u, q_1) \land \mu_2(p_2, v, q_2)\]
\[= \mu_1(p_1, uu^{-1}u, q_1) \land \mu_2(p_2, vv^{-1}v, q_2)\]
\[= (\mu_1, \mu_2)(p_1, p_2), uu^{-1}uvv^{-1}v, (q_1, q_2)\]
\[= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1, \mu_2)(p_1, p_2), u, (r_1, r_2) \land (\mu_1, \mu_2)((r_1, r_2), u^{-1}uvv^{-1}v, (q_1, q_2))\]
3.7. Cartesian product of two inverse fuzzy automata

\begin{align*}
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \cdot \mu_2)(p_1, p_2), u, (r_1, r_2) \wedge (\mu_1 \cdot \mu_2)((r_1, r_2), vv^{-1}uv^{-1}u, (q_1, q_2)) \\
&= (\mu_1 \cdot \mu_2)((p_1, p_2), uvv^{-1}uv^{-1}u, (q_1, q_2)) \\
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \cdot \mu_2)(p_1, p_2), uv^{-1}, (r_1, r_2) \wedge (\mu_1 \cdot \mu_2)((r_1, r_2), vu^{-1}u, (q_1, q_2)) \\
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \cdot \mu_2)(p_1, p_2), uv^{-1}, (r_1, r_2) \wedge (\mu_1 \cdot \mu_2)((r_1, r_2), u^{-1}uv, (q_1, q_2)) \\
&= (\mu_1 \cdot \mu_2)((p_1, p_2), wuv^{-1}u, (q_1, q_2)) \\
&= (\mu_1 \cdot \mu_2)((p_1, p_2), wuv^{-1}, (q_1, q_2)) \quad \text{for every } (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2.
\end{align*}

Similarly we can prove that\\

\begin{align*}
(\mu_1 \cdot \mu_2)((p_1, p_2), w^{-1}, (q_1, q_2)) &= (\mu_1 \cdot \mu_2)((p_1, p_2), w^{-1}ww^{-1}, (q_1, q_2)).
\end{align*}

Case 2. For \( w \in (X_1 \cup X_2)^{-1*} \) the result follows as in the above case since \((X_1 \cup X_2)^{-1*} = (X_1^{-1} \cup X_2^{-1})^*\).

Case 3. Let \( w \in ((X_1 \cup X_2) \cup (X_1 \cup X_2)^{-1})^* \). Then by the theorem there exist \( u \in (X_1 \cup X_2)^{*}, v \in (X_1 \cup X_2)^{-1*} \) such that \((\mu_1 \cdot \mu_2)((p_1, p_2), w, (q_1, q_2)) = (\mu_1 \cdot \mu_2)((p_1, p_2), uv, (q_1, q_2))\) and using case 1 and case 2 we get \( u_1 \in X_1^*, u_2 \in X_2^*, v_1 \in X_1^{-1*}, v_2 \in X_2^{-1*} \) such that \( (\mu_1 \cdot \mu_2)((p_1, p_2), w, (q_1, q_2)) \)\\

\begin{align*}
&= (\mu_1 \cdot \mu_2)((p_1, p_2), u_1 u_2 v_1 v_2, (q_1, q_2)) \\
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \cdot \mu_2)((p_1, p_2), u_1 u_2, (r_1 r_2) \wedge (\mu_1 \cdot \mu_2)((r_1, r_2), v_1 v_2, (q_1, q_2)) \\
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1(p_1, u_1, r_1) \wedge \mu_2(p_2, u_2, r_2)) \land (\mu_1(r_1, v_1, q_1) \land \mu_2(r_2, v_2, q_2)) \\
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1(p_1, u_1 u_1^{-1} u_1, r_1) \land \mu_2(p_2, u_2 u_2^{-1} u_2, r_2)) \\
&\quad \land (\mu_1(r_1, v_1 v_1^{-1} v_1, q_1) \land \mu_2(r_2, v_2 v_2^{-1} v_2, q_2) \\
&= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \cdot \mu_2)((p_1, p_2), u_1 u_1^{-1} u_1 u_2 u_2^{-1} u_2, (r_1, r_2)) \land \mu_1 \cdot \mu_2((r_1, r_2), v_1 v_1^{-1})
\end{align*}
\[ v_1 v_2 v_2^{-1} v_2, (q_1, q_2) \]

\[ = \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \mu_2)((p_1, p_2), u_1 u_2 u_2^{-1} u_1 u_2, (r_1, r_2), v_1 v_2 \]

\[ v_2^{-1} v_1 v_2, (q_1, q_2) \]

\[ = \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} (\mu_1 \mu_2)((p_1, p_2), uu^{-1} u, (r_1, r_2) \land (r_1, r_2), v_1 v_2 \]

\[ v_2^{-1} v_1 v_2, (q_1, q_2) \]

\[ = (\mu_1 \mu_2)((p_1, p_2), uu^{-1} u, (q_1, q_2)) \]

\[ = (\mu_1 \mu_2)((p_1, p_2), uu^{-1} u, (q_1, q_2)) \]

\[ = (\mu_1 \mu_2)((p_1, p_2), uu^{-1} u, (q_1, q_2)) \]

\[ = (\mu_1 \mu_2)((p_1, p_2), uu^{-1} u, (q_1, q_2)) \]

\[ \text{Similarly we can prove that} \]

\[ (\mu_1 \mu_2)((p_1, p_2), w^{-1} w, (q_1, q_2)) = (\mu_1 \mu_2)((p_1, p_2), w^{-1}, (q_1, q_2)). \]

Thus the Cartesian product \( M_1 M_2 \) is an inverse fuzzy automaton.

**Theorem 3.7.2.** Let \( M_1 = (Q_1, \bar{X}_1, \mu_1, i_1, \tau_1) \), \( M_2 = (Q_2, \bar{X}_2, \mu_2, i_2, \tau_2) \) be two fuzzy automata. If their Cartesian product \( M_1 M_2 \) is an inverse fuzzy automaton then \( M_1 \) and \( M_2 \) are inverse fuzzy automata.

**Proof.** Suppose that \( M_1 M_2 \) is an inverse fuzzy automaton. Then for every \( w \in (X_1 \cup X_2)^* \) there exist a unique \( w^{-1} \in (X_1 \cup X_2)^* \) such that

\[ \mu_1 \mu_2((p_1, p_2), w^{-1} w, (q_1, q_2)) = \mu_1 \mu_2((p_1, p_2), w, (q_1, q_2)) \]

\[ \mu_1 \mu_2((p_1, p_2), w^{-1} w, (q_1, q_2)) = \mu_1 \mu_2((p_1, p_2), w^{-1}, (q_1, q_2)) \]

for every \( (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2 \).

Let \( p, q \in Q_1 \) and \( x \in X_1^* \).

Now, \( \mu_1(p, x, q) = \mu_1 \mu_2((p, p'), x, (q, q')) \) for some \( p' = q' \in Q_2 \) and since there exist a unique \( x^{-1} \in X_1 \cap X_2^* \) such that

\[ \mu_1 \mu_2((p, p'), x, (q, q')) = \mu_1 \mu_2((p, p'), xx^{-1} x, (q, q')) = \mu_1(p, xx^{-1} x, q), \]

we get \( \mu_1(p, x, q) = \mu_1(p, xx^{-1} x, q). \)
Clearly $x^{-1} \in \tilde{X}_1^*$. Also we can prove $\mu_1(p, x^{-1}, q) = \mu_1(p, x^{-1}xx^{-1}, q)$. So $M_1$ is an inverse fuzzy automaton.

Similarly we can prove that $M_2$ is an inverse fuzzy automaton.

\[ \square \]

### 3.8 Properties of inverse fuzzy languages

A fuzzy language on an alphabet $\tilde{X}$ is inverse if and only if for every $x \in \tilde{X}^*$, $\lambda(uxx^{-1}xv) = \lambda(uxv)$ and $\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)$ for every $u, v \in \tilde{X}^*$ (Theorem 3.5.1). Let us denote the family of all inverse fuzzy languages on an alphabet $\tilde{X}$ as IFL.

**Lemma 3.8.1.** IFL is closed under quotients.

\[ \text{Proof. Let } \lambda_1, \lambda_2 \in \text{IFL. Let } x, u, v \in \tilde{X}^*. \]

Then $\lambda_1(uxx^{-1}xv) = \lambda_1(uxv)$ and $\lambda_1(ux^{-1}xx^{-1}v) = \lambda_1(ux^{-1}v)$. Then,

\[
(\lambda_2^{-1}\lambda_1)(uxx^{-1}xv) = \bigvee_{v_1 \in \tilde{X}^*} \{ \lambda_1(v_1uxx^{-1}xv) \land \lambda_2(v_1) \}
\]

\[
= \bigvee_{v_1 \in \tilde{X}^*} \{ \lambda_1(v_1uxv) \land \lambda_2(v_1) \}
\]

\[
= \bigvee_{v_1 \in \tilde{X}^*} \{ \lambda_1(v_1uxv) \land \lambda_2(v_1) \}
\]

\[
= \lambda_2^{-1}\lambda_1(uxv)
\]

and
\((\lambda_2^{-1}\lambda_1)(ux^{-1}xx^{-1}v) = \bigvee_{v_1 \in \tilde{X}^*} \{\lambda_1(v_1ux^{-1}xx^{-1}v) \land \lambda_2(v_1)\}\)
\begin{align*}
&= \bigvee_{v_1 \in \tilde{X}^*} \{\lambda_1(v_1u)x^{-1}xx^{-1}v \land \lambda_2(v_1)\} \\
&= \bigvee_{v_1 \in \tilde{X}^*} \{\lambda_1(v_1ux^{-1}v) \land \lambda_2(v_1)\} \\
&= \lambda_1^{-1}\lambda_2(ux^{-1}v).
\end{align*}

Similarly we can prove that
\begin{align*}
\lambda_1\lambda_2^{-1}(uxx^{-1}xv) &= \lambda_1\lambda_2^{-1}(uxv) \quad \text{and} \\
\lambda_1\lambda_2^{-1}(ux^{-1}xx^{-1}v) &= \lambda_1\lambda_2^{-1}(ux^{-1}v).
\end{align*}

Thus \(\lambda_1^{-1}\lambda_2 \text{ and } \lambda_2\lambda_1^{-1} \in \text{IFL}\).

\hfill \Box

**Lemma 3.8.2.** IFL is closed under multiplication by constants.

**Proof.** Let \(\lambda\) be an inverse fuzzy language on \(\tilde{X}\) and let \(x \in \tilde{X}^*\).
Then \(\lambda(uxx^{-1}xv) = \lambda(uxv)\) and \(\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)\ \forall u, v \in \tilde{X}^*\).
Let \(c \in [0, 1]\). Then
\begin{align*}
(c\lambda)(uxx^{-1}xv) &= c.\lambda(uxx^{-1}xv) = c.\lambda(uxv) = (c\lambda)(uxv) \quad \text{and} \\
(c\lambda)(ux^{-1}xx^{-1}v) &= c.\lambda(ux^{-1}xx^{-1}v) = c.\lambda(ux^{-1}v) = (c\lambda)(ux^{-1}v).
\end{align*}
Thus \(c\lambda \in \text{IFL}\).

\hfill \Box

**Theorem 3.8.1.** If \(\lambda\) is an inverse fuzzy language on \(\tilde{X}\), then \(\forall c \in [0, 1], \lambda_c\) is an inverse language on \(\tilde{X}\).

**Proof.** Let \(M = (Q, \tilde{X}, \mu, i, \tau)\) be an inverse fuzzy automaton recognizing \(\lambda\). Then for every \(x \in \tilde{X}^*\), \(\mu(p, x, q) = \mu(p, xx^{-1}x, q)\ \forall p, q \in Q\).
Let \(c \in \lambda_c\) and let \(uxv \in \lambda_c\). Then \(\lambda(uxv) \geq c\).
3.8. Properties of inverse fuzzy languages

\[ \lambda(u xv^{-1} x v) = \bigvee_{p,q \in Q} i(p) \land \mu(p, u x v^{-1} x v, q) \land \tau(q) \]
\[ = \bigvee_{p,q \in Q} i(p) \land (\bigvee_{r,r' \in Q} \mu(p, u, r) \land \mu(r, x v^{-1} x v, r') \land \mu(r', v, q)) \land \tau(q) \]
\[ = \bigvee_{p,q \in Q} i(p) \land \mu(p, u x v, q) \land \tau(q) \]
\[ = \lambda(u x v) \]

Thus \( \lambda(u x v) \geq c \) iff \( \lambda(u x v^{-1} x v) \geq c \).

ie, \( u x v \in \lambda_c \) iff \( u x x^{-1} x v \in \lambda_c \).

Similarly, we can prove that \( u x^{-1} v \in \lambda_c \) iff \( u x^{-1} x x^{-1} v \in \lambda_c \).

\[ \square \]

**Theorem 3.8.2.** *IFL is not closed under inverse homomorphic images.*

**Proof.** Let \( \tilde{X}_1 = \{a, b\}, \tilde{X}_2 = \{c, d\} \).

Let \( \beta : \tilde{X}_1 \longrightarrow \tilde{X}_2 \) defined as \( \beta(a) = \beta(b) = c \).

Then \( \beta \) can be extended to a homomorphism \( \beta^* : \tilde{X}_1^* \longrightarrow \tilde{X}_2^* \).

Let \( \lambda \) be an inverse fuzzy language on \( \tilde{X}_2 \).

Then \( \lambda(u'cv') = \lambda(u'cdcv') \) \( \forall u', v' \in \tilde{X}_2^* \).

Suppose \( \beta^{-1} \lambda \) is an inverse fuzzy language.

Then \( \beta^{-1} \lambda(uav) = \beta^{-1}(uabav) \) for all \( u, v \in \tilde{X}_1^* \).

ie, \( \lambda(\beta^*(uav)) = \lambda(\beta^*(uabav)) \).

which implies \( \lambda(\beta^*(u)c\beta^*(v)) = \lambda(\beta^*(u)ccc\beta^*(v)) \) and this says the inverse is not unique which is a contradiction. So \( \beta^{-1} \lambda \) is not an inverse fuzzy language.
Chapter 3. Regular and Inverse Fuzzy Automata

Theorem 3.8.3. IFL is not a variety of fuzzy languages.

Proof. A collection of fuzzy languages is a variety if it is closed under finite boolean operations, homomorphic and inverse homomorphic images, quotients, multiplication by constants.

An inverse fuzzy language is defined as a regular fuzzy language with its syntactic monoid is an inverse monoid. A characterization for an inverse monoid by Wagner is that a monoid is an inverse monoid iff it is regular and any two idempotents commute each other. It is also proved that a monoid is regular iff every $\mathcal{L}$-class ($\mathcal{R}$-class) contains an idempotent. Thus a fuzzy language is an inverse fuzzy language then idempotents in the syntactic monoid commute each other and every $\mathcal{L}$-class ($\mathcal{R}$-class) contains an idempotent. This property is used to prove some results on inverse fuzzy languages.

Proposition 1. Let $\lambda$ be an inverse fuzzy language on $\tilde{X}$. Let $[e]$ be an idempotent in $M(\lambda)$ then $\mu(p, xe, q) \leq \mu(p, x, q)$ and $\mu(p, ex, q) \leq \mu(p, x, q)$ for all $p, q \in Q, x \in \tilde{X}^*$.

Proof. Since $\lambda$ is an inverse fuzzy language every element of $M(\lambda)$ acts as one-one partial fuzzy transformations on $Q$ and idempotents in $M(\lambda)$ can be considered as fuzzy matrices with nonzero entries only in the diagonal and so $T_e$ acts as a subidentity on $Q$. Thus $\mu(p, e, q) \neq 0$ if $p = q$ and $= 0$ if $p \neq q$. 

Now,
\[
\mu(p, xe, q) = \bigvee_{q' \in Q} \mu(p, x, q') \land \mu(q', e, q)
\]
\[
= \mu(p, x, q) \land \mu(q, e, q)
\]
\[
\leq \mu(p, x, q).
\]
Similarly, \( \mu(p, ex, q) \leq \mu(p, x, q) \).

**Proposition 2.** If \( \lambda \) is an inverse fuzzy language then for every \( x, u, \in \tilde{X}^* \) there exists an \( n \in N \) such that \( \lambda(xu^n) \leq \lambda(x) \).

**Proof.** Since \( \lambda \) is regular \( M(\lambda) \) is finite and for \( u \in \tilde{X}^* \), \([u] \in M(\lambda) \) and since \( M(\lambda) \) is finite there exist an \( n \in N \) such that \([u]^n \) is an idempotent. And \([u]^n = [u^n] \). By the above proposition \( \mu(p, xu^n, q) \leq \mu(p, x, q) \).

Let \( i, \tau \) be the initial and final fuzzy state in the minimal automaton recognizing \( \lambda \)
\[
\lambda(xu^n) = \bigvee_{p,q \in Q} i(p) \land \mu(p, xu^n, q) \land \tau(q)
\]
\[
\leq \bigvee_{p,q \in Q} i(p) \land \mu(p, x, q) \land \tau(q)
\]
\[
= \lambda(x)
\]

**Theorem 3.8.4.** A regular fuzzy language \( \lambda \) is an inverse fuzzy language, then,

1. Idempotents of \( M(\lambda) \) commute
2. \( \forall x, u, y \in \tilde{X}^* \), there exist an \( n \in N \) such that \( \lambda(xu^ny) \leq \lambda(xy) \).

**Proof.** Let \( \lambda \) be an inverse fuzzy language.
Then (1) is obvious since $M(\lambda)$ is an inverse monoid and idempotents in an inverse monoid commute.

To prove (2), let $x, u, y \in \tilde{X}^*$. Then $[x], [u], [y] \in M(\lambda)$.

Since $M(\lambda)$ is a finite inverse monoid, there exist an $n \in N$ such that $[u]^n$ is an idempotent in $M(\lambda)$.

Let $\lambda$ be a regular fuzzy language of $\tilde{X}^*$. Let $\pi : \tilde{X}^* \rightarrow M(\lambda)$ be the syntactic morphism. Let $\lambda^+ = \{ x \in \tilde{X}^* : \lambda(x) > 0 \}$ be the support of $\lambda$. Let $\pi(\lambda^+)$ be the syntactic image of $\lambda$.

**Theorem 3.8.5.** For every regular fuzzy language the following conditions are equivalent:

1. \( \forall x, u, y \in \tilde{X}^*, \) there exist an $n \in N$ such that \( \lambda(xy) \geq \lambda(xu^ny) \).

2. \( \forall [x], [y] \in M(\lambda), \) and for every idempotent $[e] \in M(\lambda)$, $[xe] \in \pi(\lambda^+)$ implies $[xy] \in \pi(\lambda^+)$.

**Proof.** Suppose (1) is satisfied. Let $[x], [e], [y] \in M(\lambda)$ such that $[xe] \in \pi(\lambda^+)$. Since $\pi$ is onto, there exist $x, u, y \in \tilde{X}^*$ such that $\pi(x) = [x]$, $\pi(y) = [y]$, $\pi(u) = [e]$.

By (a), there exist an $n \in N$ such that $\lambda(xy) \geq \lambda(xu^ny)$. 

Let $\lambda$ be a regular fuzzy language of $\tilde{X}^*$. Let $\pi : \tilde{X}^* \rightarrow M(\lambda)$ be the syntactic morphism. Let $\lambda^+ = \{ x \in \tilde{X}^* : \lambda(x) > 0 \}$ be the support of $\lambda$. Let $\pi(\lambda^+)$ be the syntactic image of $\lambda$. 

**Theorem 3.8.5.** For every regular fuzzy language the following conditions are equivalent:

1. \( \forall x, u, y \in \tilde{X}^*, \) there exist an $n \in N$ such that $\lambda(xy) \geq \lambda(xu^ny)$.

2. \( \forall [x], [y] \in M(\lambda), \) and for every idempotent $[e] \in M(\lambda)$, $[xe] \in \pi(\lambda^+)$ implies $[xy] \in \pi(\lambda^+)$.

**Proof.** Suppose (1) is satisfied. Let $[x], [e], [y] \in M(\lambda)$ such that $[xe] \in \pi(\lambda^+)$. Since $\pi$ is onto, there exist $x, u, y \in \tilde{X}^*$ such that $\pi(x) = [x]$, $\pi(y) = [y]$, $\pi(u) = [e]$.

By (a), there exist an $n \in N$ such that $\lambda(xy) \geq \lambda(xu^ny)$. 

Let $\lambda$ be a regular fuzzy language of $\tilde{X}^*$. Let $\pi : \tilde{X}^* \rightarrow M(\lambda)$ be the syntactic morphism. Let $\lambda^+ = \{ x \in \tilde{X}^* : \lambda(x) > 0 \}$ be the support of $\lambda$. Let $\pi(\lambda^+)$ be the syntactic image of $\lambda$. 

**Theorem 3.8.5.** For every regular fuzzy language the following conditions are equivalent:

1. \( \forall x, u, y \in \tilde{X}^*, \) there exist an $n \in N$ such that $\lambda(xy) \geq \lambda(xu^ny)$.

2. \( \forall [x], [y] \in M(\lambda), \) and for every idempotent $[e] \in M(\lambda)$, $[xe] \in \pi(\lambda^+)$ implies $[xy] \in \pi(\lambda^+)$.

**Proof.** Suppose (1) is satisfied. Let $[x], [e], [y] \in M(\lambda)$ such that $[xe] \in \pi(\lambda^+)$. Since $\pi$ is onto, there exist $x, u, y \in \tilde{X}^*$ such that $\pi(x) = [x]$, $\pi(y) = [y]$, $\pi(u) = [e]$.

By (a), there exist an $n \in N$ such that $\lambda(xy) \geq \lambda(xu^ny)$.
\[ \pi(xu^n y) = \pi(x)\pi(u^n)\pi(y) = [x][e]^n[y] = [x][e][y] \in \pi(\lambda^+) \] by assumption. So \( xu^n y \in \lambda^+ \) and since \( \lambda(xy) \geq \lambda(xu^n y) \), \( xy \in \lambda^+ \).

So \( [x][y] = [xy] \in \pi(\lambda^+) \).

Conversely, suppose that for every \([x], [y] \in M(\lambda)\) and for every idempotent \([e] \in M(\lambda)\), \([xey] \in \pi(\lambda^+) \implies [xy] \in \pi(\lambda^+) \)

When \( \lambda(xu^n y) = 0 \) for all \( n \in N \), the result is clear. Suppose \( x, u, y \in \tilde{X}^* \) such that \( \lambda(xu^n y) > 0 \) for all \( n \in N \).

ie, \( xu^n y \in \lambda^+ \). Then \([x], [y], [u] \in M(\lambda)\) and since \( M(\lambda) \) is finite there exist a \( k \in N \) such that \([u]^k\) is an idempotent say \([e]\).

Now, \([xey] = [x][e][y] = \pi(x)\pi(u^k)\pi(y) = \pi(xu^k y) \in \pi(\lambda^+) \).

So \([xy] \in \pi(\lambda^+) \) by the assumption. ie, \( \pi(xy) \in \pi(\lambda^+) \) and this implies \( xy \in \lambda^+ \). ie, \( \lambda(xy) > 0 \) which says that there exist a \( k \in N \) such that \( \lambda(xy) \geq \lambda(xu^k y) \).

Thus we have proved the theorem,

**Theorem 3.8.6.** A regular fuzzy language \( \lambda \) is an inverse fuzzy language, then,

1. Idempotents of \( M(\lambda) \) commute
2. For every \([x], [y] \in M(\lambda)\), and for every idempotent \([e] \in M(\lambda)\), \([xey] \in \pi(\lambda^+) \) implies \([xy] \in \pi(\lambda^+) \).

**Proof.** From theorems 3.8.4 and 3.8.5 we get 3.8.6. \( \square \)

**Theorem 3.8.7.** Let \( \lambda \) is an inverse fuzzy language. Then

1. Idempotents of \( M(\lambda) \) commute.
(2) For every \( x, y \in \tilde{X}^* \), there exist an idempotent \([e] \in M(\lambda)\) such that
\[ [xey] \in \pi(\lambda^+) \text{ if and only if } [xy] \in \pi(\lambda^+) \].

Proof. Since the syntactic monoid of an inverse fuzzy language is an inverse monoid, (1) is obvious.

For (2), let \( x \in \tilde{X}^* \). Since \( M(\lambda) \) is an inverse monoid, \( \mathcal{L}_x \) contains a unique idempotent say \([e]\) which is a right identity for elements of \( \mathcal{L}_x \).

Then \( T_x \circ T_e = T_x \). Let \((Q, \tilde{X}, \mu, i, \tau)\) be the minimal fuzzy automata recognizing \( \lambda \). Suppose \( y \in \tilde{X}^* \) such that \([xey] \in \pi(\lambda^+)\).

Now, \( \lambda(xey) = i \circ T_x \circ T_e \circ T_y \circ \tau \)
\[ = i \circ T_x \circ T_y \circ \tau \]
\[ = \lambda(xy) \).

So \( \lambda(xey) > 0 \iff \lambda(xy) > 0 \)

ie, \([xey] \in \pi(\lambda^+)\) if and only if \([xy] \in \pi(\lambda^+)\).

\( \square \)

**Theorem 3.8.8.** Let \( M = (Q, \tilde{X}, \mu) \) be a fuzzy automaton. Then for every \( x, y \in \tilde{X}^* \) and \( m, n \in \mathbb{N} \) with \([x]^m, [y]^n\) are idempotents in \( \tilde{X}^*/\theta_M \),
\( \mu(p, x^m y^n, q) = \mu(p, y^n x^m, q) \) for every \( p, q \in Q \) if and only \( \tilde{X}^*/\theta_M \) has commuting idempotents.

Proof. Since \( \tilde{X}^*/\theta_M \) is a finite semigroup, for every \([x]\) in \( \tilde{X}^*/\theta_M \) there exists an \( n \in \mathbb{N} \) such that \([x]^n\) is an idempotent. Let \([x]^m, [y]^n\) be two idempotents in \( \tilde{X}^*/\theta_M \). Now \( \mu(p, x^m y^n, q) = \mu(p, y^n x^m, q) \) for every \( p, q \in Q \) iff \([x^m y^n] = [y^n x^m]\) iff \([x]^m[y]^n = [y^n][x]^m\) iff \([x]^m[y]^n = [y]^n[x]^m\) iff \( \tilde{X}^*/\theta_M \) has commuting idempotents.
3.8. Properties of inverse fuzzy languages

**Theorem 3.8.9.** If $\lambda$ is a fuzzy language. Then for every $x, y \in \tilde{X}^*$ and $m, n \in N$ such that $[x]^m, [y]^n$ are idempotents, $\lambda(ux^my^nv) = \lambda(uy^nx^mv) \forall u, v \in \tilde{X}^*$, if and only if the syntactic monoid of $\lambda$ has commuting idempotents.

**Proof.** Since for every $x, y \in \tilde{X}^*$ and $m, n \in N$ such that $[x]^m, [y]^n$ are idempotents, $\lambda(ux^my^nv) = \lambda(uy^nx^mv)$, $x^my^n p_\lambda y^nx^m$ iff $[x]^m[y]^n_{p_\lambda} = [y]^n[x]^m_{p_\lambda}$ iff syntactic monoid of $\lambda$ has commuting idempotents.

Thus we have proved the theorem.

**Theorem 3.8.10.** If $\lambda$ is an inverse fuzzy language, then,

1. for every $x, y \in \tilde{X}^*$ there exist $m, n \in N$ such that
   $\lambda(ux^my^nv) = \lambda(uy^nx^mv) \forall u, v \in \tilde{X}^*$.

2. $\forall x, u, y \in \tilde{X}^*$, there exist an $n \in N$ such that $\lambda(xu^ny) \leq \lambda(xy)$.

By Eilenberg-type variety theorem, the collection of all fuzzy languages such that for every $x, y \in \tilde{X}^*$ and $m, n \in N$ such that $[x]^m, [y]^n$ are idempotents, $\lambda(ux^my^nv) = \lambda(uy^nx^mv) \forall u, v \in \tilde{X}^*$, form a variety of fuzzy languages and the associated pseudo-variety is the variety generated by finite inverse monoids.