CHAPTER V.

FIXED POINT THEOREMS
FOR MAPPINGS WITH VARIABLE COEFFICIENTS
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WITH VARIABLE COEFFICIENTS

5.1 In 1995, Nikola Jotic ([1], Theorem 1) established the following important results:

THEOREM A Let \((X, d)\) be a complete metric space, \(f: X \to X\) a self-mapping and \(\phi: [0, \infty) \to [0, \infty)\) a real function such that

\[
\begin{align*}
(5.1.1) & \quad \phi(r) < r \text{ for } r > 0, \\
(5.1.2) & \quad \lim_{r \to r} \sup \phi(t) < r \text{ for } r > 0.
\end{align*}
\]

If \(f: X \to X\) satisfies the condition

\[
(5.1.3) \quad d[f(x), f^2(x)] \leq \phi[d(x, f(x))]
\]

for every \(x \in X\), then \(\{f^n(x)\}\) is a Cauchy sequence.

With the help of above theorem, we now generalize some results of Jaggi [2], Dass and Gupta [1], Chatterjee [1] and other known generalizations of Banach contraction principle.

THEOREM 1 Let \((X, d)\) be complete metric space and \(f\) a self-mapping of \(X\). If there exists real functions \(\alpha, \beta, \gamma, \delta: (0, \infty) \to (-\infty, +\infty)\) which are continuous from the right and such that for each \(x, y \in X\) and \(r = d(x, y)\) the following inequalities hold:

\[
(5.1.4) \quad \alpha(r) d(x, y) + \beta(r) d[f(x), f(y)] + \gamma(r) [d(x, f(y)) + d(y, f(x))]
\]
\[ + \delta(r) \frac{d(y, f(y)) [1 + d(x, f(x))]}{1 + d(x, y)} \geq 0 \]

(5.1.5) \[ \alpha(r) + \beta(r) + \gamma(r) + |\gamma(r)| + \delta(r) < 0, \]
(5.1.6) \[ \beta(r) + \gamma(r) + \delta(r) < 0, \]
(5.1.7) \[ \alpha(r) + \beta(r) + 2\gamma(r) < 0, \]

then \( f \) has a unique fixed point in \( X \).

**Proof** Let \( x \in X \) be arbitrary and let \( y = f(x) \). Put \( d_x = d(x, y) = d[x, f(x)] \).

Then applying (5.1.4), we have

(5.1.8) \[ \alpha(d_x)d_x + \beta(d_x) d(f(x), f^2(x) + \gamma(d_x) d(x, f(x)) + \delta(d_x) \} (f(x), f^2(x) \geq 0 \]

We shall consider two cases:

**Case 1.** \( \gamma(d_x) \geq 0 \). By the triangle inequality and \( \gamma(d_x) \geq 0 \) we get

\[ \gamma(d_x) d(x, f(x)) \leq \gamma(d_x) d[x, f(x)] + \gamma(d_x) d[f(x), f^2(x)] \]

\[ = \gamma(d_x)d_x + \gamma(d_x) d[f(x), f^2(x)] \]

Then by (5.1.8) we have

\[ [\alpha(d_x) + \gamma(d_x)]d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)] d[f(x), f^2(x)] \geq 0 \]

and hence

(5.1.9) \[ [\alpha(d_x) + \gamma(d_x)]d_x \geq -[\beta(d_x) + \gamma(d_x) + \delta(d_x)] d[f(x), f^2(x)] \]

From this inequality and (5.1.6) it follows that as

\[ \gamma(d_x) = |\gamma(d_x)| \]

(5.1.10) \[ \alpha(d_x) + |\gamma(d_x)| \geq 0. \]

So from (5.1.9) we have

(5.1.11) \[ d[f(x), f^2(x)] \leq - \frac{\alpha(d_x) + |\gamma(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} d_x. \]

From (5.1.5) for \( \gamma(d_x) \geq 0 \) we have
\[
[\alpha(d_x) + \gamma(d_x)] < - [\beta(d_x) + \gamma(d_x) + \delta(d_x)]
\]

Now (5.1.6) and (5.1.10) implies

\[(5.1.12) \quad 0 \leq -\frac{\alpha(d_x) + |\gamma(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} < 1.\]

Define a function \(\phi: [0, \infty) \to [0, \infty)\) as follows that

\[(5.1.13) \quad \phi(t) = -\frac{\alpha(t) + |\gamma(t)|}{\beta(t) + \gamma(t) + \delta(t)} t\]

Then from (5.1.11) and (5.1.12) we have, for all \(x \in X\)

\[(5.1.14) \quad d[f(x), f^2(x)] \leq \phi(d(x, f(x))).\]

Case 2. Suppose now that \(\gamma(d_x) < 0\). By the triangle inequality

\[d[x, f^2(x)] \geq d[f(x), f^2(x)] - d[x, f(x)]\]

and since \(\gamma(d_x) < 0\) we have

\[\gamma(d_x) d[x, f^2(x)] \leq \gamma(d_x) d[f(x), f^2(x)] - \gamma(d_x) d[x, f(x)]\]

Now from (5.1.8) we obtain

\[\alpha(d_x) d_x + \beta(d_x) d[f(x), f^2(x)] + \gamma(d_x) [d[f(x), f^2(x)] - d_x] + \delta(d_x) d[f(x), f^2(x)] \geq 0\]

and hence

\[\alpha(d_x) - \gamma(d_x)] d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)] d[f(x), f^2(x)] \geq 0\]

Hence, using (5.1.6) and as \(-\gamma(d_x) = |\gamma(d_x)|\), we get

\[d[f(x), f^2(x)] \leq -\frac{\alpha(d_x) + |\gamma(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} d_x\]

From \(\gamma(d_x) < 0\) and (5.1.5) it follows that

\[\alpha(d_x) + |\gamma(d_x)| + \beta(d_x) + \gamma(d_x) + \delta(d_x) < 0.\]

Hence we again obtain the relation (5.1.12). If we now, define a function \(\phi: [0, \infty] \to [0, \infty]\), as before by formula (5.1.13), we obtain the relation
Therefore, in both cases $\gamma(dx) \geq 0$ and $\gamma(dx) < 0$ we obtain the relation (5.1.14), where $\phi$ is defined by (5.1.13).

Take $x_0 = x$ and define sequence $\{x_n\}_{n \in N}$ by $x_n = f(x_{n-1})$.

From (5.1.12) it follows that the function $\phi(t)$ defined by (5.1.13) satisfies (5.1.1). Since $\alpha, \beta, \gamma$ and $\delta$ are continuous from the right and from (5.1.6) $\beta(r) + \gamma(r) + \delta(r) \neq 0$, we conclude that

$$\lim_{t \to r^-} \phi(t) = -\frac{\alpha(r) + |\gamma(r)|}{\beta(r) + \gamma(r) + \delta(r)} r = \phi(r)$$

so $\phi(t)$ is continuous from the right. Therefore, for $r > 0$ we have $\lim_{t \to r^-} \phi(t) = \phi(r) < r$, and thus the function $\phi(t)$ satisfies (5.1.2). From (5.1.14) $\phi(t)$ also satisfies (5.1.3). Therefore, from Theorem A it follows that $\{x_n\}$ is a Cauchy sequence which is convergent in $X$ as $X$ is complete.

Now, we shall prove that $\lim_{n \to \infty} x_n = x^*$ implies $x^* = f(x^*)$. Suppose $d_n = d(x_n, x^*)$ and then (5.1.4) we have

$$\alpha(d_n) d(x_n, x^*) + \beta(d_n) d(f(x_n), f(x^*)) + \gamma(d_n) |d(x_n, f(x_n)| + d(x^*, f(x_n))$$

$$+ \delta(d_n) \frac{d[x^*, f(x^*)][1 + d[x_n, f(x_n)]}{1 + d(x_n, x^*)} \geq 0$$

this inequality gives

$$\alpha(d_n) d(x_n, x^*) + \beta(d_n) d(x_{n+1}, f(x^*)) + \gamma(d_n) |d(x_n, f(x_n)| + d(x^*, x_{n+1}))$$

$$+ \delta(d_n) \frac{d[x^*, f(x^*)][1 + d[x_n, x_{n+1}]}{1 + d(x_n, x^*)} \geq 0$$

Letting $n$ tend to infinity, we get
[\beta(0) + \gamma(0) + \delta(0)] d[x^*, f(x^*)] \geq 0

because \( \alpha, \beta, \gamma, \delta \) are continuous from the right. By this inequality and

(5.1.6) it follows that \( d[x^*, f(x^*)]/ = 0 \). Hence \( x^* = f(x^*) \).

Further to prove the uniqueness of this fixed point. Let \( y_1 = f(y_1), y_2 = f(y_2) \)

and \( r = d(y_1, y_2) \). From (5.1.4) it follows that

\[
\alpha (r) d(y_1, y_2) + \beta (r) d(f(y_1), f(y_2)) + \gamma (r) [d(y_1, f(y_2)] + d(y_2, f(y_1))] +
\]

\[
+ \delta (r) \frac{d[y_2, f(y_2)] + d[y_1, f(y_1)]}}{1 + d(y_1, y_2)} \geq 0
\]

which implies

\[
[\alpha (r) + \beta (r) + 2\gamma (r)] d(y_1, y_2) \geq 0.
\]

By (5.1.7) it follows that \( d(y_1, y_2) = 0 \) i.e. \( y_1 = y_2 \) which completes the proof.

**COROLLARY 1.** Let \((X, d)\) be a complete metric space and \( f \) a self-mappings

of \( X \). If \( \alpha, \gamma, \delta: (0, + \infty) \to (-\infty, + \infty) \) are functions as in Theorem 1 and

such that the inequalities (5.1.4), (5.1.5), (5.1.6) and (5.1.7) are satisfied with

\( \beta = -1 \), then \( f \) has a unique fixed point in \( X \).

**REMARKS.** (1) If in corollary 1, \( \alpha, \gamma \) are non-negative constants and \( \delta = 0 \), it

reduce to a Theorem due to Dass and Gupta \{[1], Theorem 1\}.

(2) If in corollary 1, \( \gamma \) is non-negative constant and \( \alpha = \delta = 0 \), we get a

result of Chatterjee [1].

5.2 We now extend a result of Jotic \{[1], Theorem 2\} which includes several

interesting results due to 'Ciric' [1], Hardy and rogers [1], Hicks and Rhoades

[1], Ivanov [1], Jaggi and Dass [1], Pal and Maiti [1] and Zamfirescu [1].

**THEOREM 2.** Let \((X, d)\) be complete metric space and \( f \) a self-mapping of
If there exists real functions \( \alpha, \beta, \gamma, \delta, \eta : (0, \infty) \to (-\infty, +\infty) \) which are continuous from the right and such that for all distinct \( x, y \in X \) and \( r = d(x, y) \) the following inequalities hold:

\[
(5.2.1) \quad \alpha(r) d(x, y) + \beta(r) d[f(x), f(y)] + \gamma(r) \{d[x, f(x)] + d[y, f(y)]\} + \delta(r)[d(x, f(y)) + d(y, f(x))] + \eta(r) \frac{d[x, f(x)] d[y, f(y)]}{d[x, f(y)] + d[y, f(x)] + d(x, y)} \geq 0
\]

\[
(5.2.2) \quad \alpha(r) + \beta(r) + 2\gamma(r) + \delta(r) + \eta(r) < 0.
\]

\[
(5.2.3) \quad \beta(r) + \gamma(r) + \delta(r) < 0
\]

\[
(5.2.4) \quad \alpha(r) + \beta(r) + 2\delta(r) < 0
\]

then \( f \) has a unique fixed point in \( X \).

**Proof.** Let \( x \in X \) be arbitrary and let \( y = f(x) \). Put \( d_x = d(x, y) = d[x, f(x)] \).

Then by (5.2.1) we obtain

\[
(5.2.5) \quad \alpha(d_x) d_x + \beta(d_x) d[f(x), f^2(x)] + \gamma(d_x) d[x, f(x)] + \delta(d_x) d[f(x), f^2(x)]
\]

\[
+ \delta(d_x) d[x, f^2(x)] + \eta(d_x) d_x \geq 0
\]

We shall consider two cases:

**Case 1.** \( \delta(d_x) \geq 0 \). By the triangle inequality and \( \delta(d_x) \geq 0 \), we have

\[
\delta(d_x) d[x, f^2(x)] \geq \delta(d_x) d[x, f(x)] + \delta(d_x) d[f(x), f^2(x)]
\]

\[
= \delta(d_x) d_x + \delta(d_x) d[f(x), f^2(x)]
\]

Then from (5.2.5) gives

\[
[\alpha(d_x) + \gamma(d_x) + \delta(d_x)] d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)] d[f(x), f^2(x)] \geq 0
\]

and hence

\[
(5.2.6) \quad [\alpha(d_x) + \gamma(d_x) + \delta(d_x)] d_x \geq -[\beta(d_x) + \gamma(d_x) + \delta(d_x)] d[f(x), f^2(x)]
\]
From this inequality and (5.2.3) it follows that, as \( \delta(d_x) = |\delta(d_x)| \)

(5.2.7) \[ \alpha(d_x) + \gamma(d_x) + |\delta(d_x)| + \eta(d_x) \geq 0, \]

so from (5.2.6) we have

(5.2.8) \[ d[f(x), f^2(x)] \leq \frac{-\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| + \eta(d_x)}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} d_x \]

From (5.2.2) for \( \delta(d_x) \geq 0 \) we have

\[ [\alpha(d_x) + \gamma(d_x) + \delta(d_x) + \eta(d_x)] < [\beta(d_x) + \gamma(d_x) + \delta(d_x)] \]

Now (5.2.3) and (5.2.7) yields

(5.2.9) \[ 0 \leq \frac{-\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| + \eta(d_x)}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} < 1 \]

Define a function \( \phi : [0, \infty) \to [0, \infty) \) as follows

(5.2.10) \[ \phi(t) = \frac{-\alpha(t) + \gamma(t) + |\delta(t)| + \eta(t)}{\beta(t) + \gamma(t) + \delta(t)} t. \]

Then from (5.2.8) and (5.2.9) we have for all \( x \in X \)

(5.2.11) \[ d[f(x), f^2(x)] \leq \phi([x, f(x)]). \]

**Case 2.** Let \( \delta(d_x) \leq 0 \). Then the triangle inequality gives

\[ d[x, f^2(x)] \geq d[f(x), f^2(x)] - d[x, f(x)] \]

and since \( \delta(d_x) < 0 \) we have

\[ \delta(d_x) d[x, f^2(x)] \leq \delta(d_x) d[f(x), f^2(x)] - \delta(d_x) d[x, f(x)] \]

Further, from (5.2.5) we get

\[ \alpha(d_x) d_x + \beta(d_x) d[f(x), f^2(x)] + \gamma(d_x) [d_x + d[f(x), f^2(x)]] \]

\[ + \delta(d_x) [d[f(x), f^2(x)] - d_x] + \eta(d_x) d_x \geq 0. \]
and hence
\[
[\alpha(d_x) + \gamma(d_x) + \eta(d_x) - \delta(d_x)] dx + [\beta(d_x) + \gamma(d_x) + \delta(d_x)] \geq 0
\]

Hence, using (5.2.3) and as \(-\delta(d_x) = |\delta(d_x)|\), we get
\[
d[f(x), f^2(x)] \leq -\frac{\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| + \eta(d_x)}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} dx
\]

From \(\delta(d_x) < 0\) and (5.2.2) it follows that
\[
\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| + \eta(d_x) + \beta(d_x) + \gamma(d_x) + \delta(d_x) < 0
\]

Hence we again obtain the relation (5.2.9). If we now, define a function
\[
\phi : [0, \infty) \to [0, \infty), \text{ as before by formula } (5.2.10), \text{ we obtain the relation (5.2.11). Therefore, in both cases } \delta(d_x) \geq 0 \text{ and } \delta(d_x) < 0 \text{ we have the relation (5.2.11), where } \phi \text{ is defined by (5.2.10).}
\]

Take \(x_0 = x\) and define sequence \(\{x_n\}_{n \in \mathbb{N}}\) by \(x_n = f(x_{n-1})\).

From (5.2.9) it follows that the function \(\phi(t)\) defined by (5.2.10) satisfies (5.1.1). Since \(\alpha, \beta, \gamma, \delta\) and \(\eta\) are continuous from the right and from (5.2.3) \(\beta(r) + \gamma(r) + \delta(r) \neq 0\), we conclude that
\[
\lim_{t \to r^+} \phi(t) = -\frac{\alpha(r) + \gamma(r) + |\delta(r)| + \eta(r)}{\beta(r) + \gamma(r) + \delta(r)} r = \phi(r)
\]
so \(\phi(t)\) is continuous from the right. Therefore, for \(r > 0\) we have
\[
\lim_{t \to r^+} \sup \phi(t) = \lim_{t \to r^+} \phi(t) = \phi(r) < r
\]
and thus the function \(\phi(t)\) satisfies (5.1.2). From (5.2.11) \(\phi(t)\) also satisfies (5.1.3). Therefore, from Theorem A it follows that \(\{x_n\}\) is a Cauchy sequence which is convergent in \(X\) as \(X\) is complete.
Now we shall prove that \( \lim_{n \to \infty} x_n = x^* \) implies \( x^* = f(x^*) \). Suppose 
\[
d_n = d(x_n, x^*)
\]
and then by (5.2.1) we have
\[
\alpha(d_n)d(x_n, x^*) + \beta(d_n)d(f(x_n), f(x^*)) + \gamma(d_n)\left[d(x_n, f(x_n)) + d(x^*, f(x^*))\right] \\
\delta(d_n)\left[d(x_n, f(x^*)) + d(x^*, f(x_n))\right] \\
\eta(d_n)\frac{d\left[x_n, f(x^*)\right] + d\left[x^*, f(x_n)\right]}{d\left[x_n, f(x^*)\right] + d\left[x^*, f(x_n)\right] + d(x_n, x^*)} \geq 0
\]
This inequality gives
\[
\alpha(d_n)d(x_n, x^*) + \beta(d_n)d(x_{n+1}, f(x^*)) + \gamma(d_n)\left[d(x_n, x_{n+1}) + d(x^*, f(x^*))\right] \\
\delta(d_n)\left[d(x_n, f(x^*)) + d(x^*, x_{n+1})\right] \\
\eta(d_n)\frac{d(x_n, x_{n+1})d(x^*, f(x^*))}{d(x_n, f(x^*)) + d(x^*, x_{n+1}) + d(x_n, x^*)} \geq 0
\]
Letting \( n \) tend to infinity, we get
\[
[\beta(0) + \gamma(0) + \delta(0)]d\left[x^*, f(x)\right] \geq 0
\]
because \( \alpha, \beta, \gamma, \delta, \) and \( \eta \) are continuous from the right. By this inequality and (5.2.3) it follows that \( d[x^*, f(x^*)] = 0 \). Hence \( x^* = f(x^*) \).

The uniqueness of this fixed point can be easily obtained as in theorem 1.

**COROLLARY 2.** Let \( (X, d) \) be a complete metric space and \( f : X \to X \) a self-mapping. If \( \alpha, \gamma, \delta, \eta : (0, +\infty) \to (-\infty, +\infty) \) are functions as in Theorem 2 and such that the inequalities (5.2.1), (5.2.2), (5.2.3) and (5.2.4) are satisfied with \( \beta = -1 \), then \( f \) has a unique fixed point in \( X \).

**REMARKS**

(3) If in corollary 2, functions \( \alpha, \gamma, \delta \) are non-negative and \( \eta = 0 \), then it reduces to a theorem which contains a result of Hardy and Rogers ([1], Theorem 2.)
(4) If in corollary 2, \( \alpha, \delta \) and \( \eta \) are constants and \( \eta = 0 \), then it includes Theorem of Ciric [1], Zamfirescu [1] and Hardy and Rogers ([1], Theorem 1).

(5) If in Theorem 2, we take \( \eta = 0 \), we get a result of Jotic ([1], Theorem 1).

(6) If in Theorem 2, \( \alpha, \eta \) are non-negative constants, \( \gamma = \delta = 0 \) and \( \beta = -1 \), it reduces to a result of Jaggi and Dass [1].

**THEOREM 3.** Let \((X, d)\) be a complete metric space and \( f \) a self-mapping of \( X \). If there exists a real functions \( \alpha, \beta, \gamma, \delta : (0, \infty) \to (-\infty, \infty) \) which are continuous from the right and such that for each \( x, y \in X \) for which \( d(x, fy) + d(y, fx) \neq 0 \), and \( r = d(x, y) \) the following inequalities hold:

\[
\alpha(r)d(x, y) + \beta(r)d[f(x), f(y)] + \gamma(r) \max \{d[x, f(y)], d[y, f(x)]\} + \delta(r) \left[ \frac{d[x, f(x)]d[x, f(y)] + d[y, f(y)]d[y, f(x)]}{d[x, f(x)] + d[y, f(y)]} \right] \geq 0
\]

(5.2.12)

(5.2.13) \( \alpha(r) + \beta(r) + \gamma(r) + |\gamma(r)| + \delta(r) < 0 \),

(5.2.14) \( \beta(r) + \gamma(r) < 0 \),

(5.2.15) \( \alpha(r) + \beta(r) + \gamma(r) < 0 \)

then \( f \) has a unique fixed point in \( X \).

**PROOF** Let \( x \in X \) be arbitrary and let \( y = f(x) \). Put \( d_x = d(x, y) = d[x, f(x)] \).

Then by (5.2.12) we have

\[
\alpha(d_x)d_x + \beta(d_x) d[f(x), f^2(x)] + \gamma(d_x) d[x, f^2(x)] + \delta(d_x)d_x \geq 0
\]

(5.2.16)

We shall consider two cases:

**Case 1.** \( \gamma(d_x) \geq 0 \). By the triangle inequality and \( \gamma(d_x) \geq 0 \) we get

\[
\gamma(d_x) d[x, f^2(x)] \leq \gamma(d_x) d[x, f(x)] + \gamma(d_x) d[f(x), f^2(x)]
\]

\[
= \gamma(d_x) d_x + \gamma(d_x) d[f(x), f^2(x)]
\]
Then by (5.2.16) we have
\[
[\alpha(d_x) + \gamma(d_x) + \delta(d_x)] d x + [\beta(d_x) + \gamma(d_x)] d[f(x), f^2(x)] \geq 0
\]
and hence
\[
\alpha(d_x) + \gamma(d_x) + \delta(d_x) \geq -[\beta(d_x) + \gamma(d_x)] d[f(x), f^2(x)]
\]
from this inequality and (5.2.14) it follows that, as \(\gamma(d_x) = \frac{1}{2} \gamma(d_x)\),
\[
\alpha(d_x) + \gamma(d_x) + \delta(d_x) \geq 0
\]
So from (5.2.17) we have
\[
d[f(x), f^2(x)] \leq -\frac{\alpha(d_x) + \gamma(d_x) + \delta(d_x)}{\beta(d_x) + \gamma(d_x)} d x
\]
From (5.2.13) for \(\gamma(d_x) \geq 0\) we have
\[
[\alpha(d_x) + \gamma(d_x) + \delta(d_x)] < -[\beta(d_x) + \gamma(d_x)]
\]
Now (5.2.14) and (5.2.18) implies
\[
0 \leq -\frac{\alpha(d_x) + \gamma(d_x) + \delta(d_x)}{\beta(d_x) + \gamma(d_x)} < 1
\]
Define a function \(\phi: [0, \infty) \to [0, \infty)\) as follows that
\[
\phi(t) = -\frac{\alpha(t) + \gamma(t) + \delta(t)}{\beta(t) + \gamma(t)}
\]
Then from (5.2.19) and (5.2.20) we have, for all \(x \in X\)
\[
d[f(x), f^2(x)] \leq \phi[d(x), f(x)].
\]
**Case 2.** Suppose now that \(\gamma(dx) < 0\). By the triangle inequality
\[
d[x, f^2(x)] \geq d[f(x), f^2(x)] - d[x, f(x)]
\]
and since \(\gamma(dx) < 0\) we get
\[
\gamma(dx) d[x, f^2(x)] \leq \gamma(dx) d[f(x), f^2(x)] - \gamma(dx) dx
\]
Now from (5.2.16) we obtain
\[ \alpha(d_x) dx + \beta(d_x) \, df(x) \, f^2(x) + \gamma(d_x) \, \{ df(x), f^2(x) \} - d_x + \delta(d_x) dx \geq 0 \]

and hence

\[ [\alpha(d_x) - \gamma(d_x) + \delta(d_x)] dx + [\beta(d_x) + \gamma(d_x)] \, df(x), f^2(x) \geq 0 \]

Hence, using (5.2.14) and as \( -\gamma(d_x) = |\gamma(d_x)| \), we get

\[ d[f(x), f^2(x)] \leq \frac{\alpha(d_x) + |\gamma(d_x)| + \delta(d_x)}{\beta(d_x) + \gamma(d_x)} \, dx \]

From \( \gamma(d_x) < 0 \) and (5.2.13) it follows that

\[ \alpha(d_x) + |\gamma(d_x)| + \delta(d_x) - \beta(d_x) + \gamma(d_x) < 0 \]

Hence we again obtain the relation (5.2.20). If we now, define a function \( \phi: [0, \infty) \rightarrow [0, \infty) \), as before by formula (5.2.21), we obtain the relation (5.2.22). Therefore, in both cases \( \gamma(d_x) \geq 0 \) and \( \gamma(d_x) < 0 \) we obtain the relation (5.2.22), where \( \Phi \) is defined by (5.2.21).

Take \( x_0 = x \) and define the sequence \( \{ x_n \}_{n \in \mathbb{N}} \) by \( x_n = f(x_{n-1}) \).

From (5.2.20) we conclude that the function \( \Phi(t) \) defined by (5.2.21) satisfy (5.1.1). Since \( \alpha, \beta, \gamma \) and \( \delta \) are continuous from the right and from (5.2.14) \( \beta(r) + \gamma(r) \neq 0 \), it follows that

\[ \lim_{t \to r^+} \Phi(t) = -\frac{\alpha(r) + |\gamma(r)| + \delta(r)}{\beta(r) + \gamma(r)} \, r = \Phi(r) \]

so \( \Phi(t) \) is continuous from the right. Therefore for \( r > 0 \), we have

\[ \limsup_{t \to r^+} \Phi(t) = \lim_{t \to r^+} \Phi(t) = \Phi(r) < r \]

and we conclude that the function \( \Phi(t) \) satisfies (5.1.2). From (5.2.22), \( \Phi(t) \) also satisfies (5.1.3). Therefore from Theorem A we conclude that \( \{ x_n \} \) is a Cauchy sequence which is convergent as \( X \) is complete.
Now we shall prove that $\lim_{n \to \infty} x_n = x^*$ implies $x^* = f(x^*)$. Let $d_n = d(x_n, x^*)$.

By (5.2.12) we have

$$\alpha(d_n)d(x_n, x^*) + \beta(d_n)d(f(x_n), f(x^*))$$

$$+ \gamma(d_n)\max\left[d(x_n, f(x^*)), d(x^*, f(x_n))\right]$$

$$+ \delta(d_n)\left[\frac{d(f(x_n), x_n)d(f(x^*), x_n) + d(f(x^*), x^*)d(f(x_n), x^*)}{d(f(x_n), x_n) + d(f(x^*), x_n)}\right] \geq 0$$

This inequality gives

$$\alpha(d_n)d(x_n, x^*) + \beta(d_n)d(x_{n+1}, f(x^*))$$

$$+ \gamma(d_n)\max\left[d(x_n, f(x^*)), d(x^*, x_{n+1})\right]$$

$$+ \delta(d_n)\left[\frac{d(x_{n+1}, x_n)d(x_n, f(x^*)) + d(f(x^*), x^*)d(x^*, x_{n+1})}{d(x_{n+1}, x_n) + d(f(x^*), x_n)}\right] \geq 0$$

Letting $n$ tend to infinity, we get

$$(\beta(0) + \gamma(0))d(f(x^*), x^*) \geq 0$$

because $\alpha, \beta, \gamma, \delta$ are continuous from the right. By this inequality and (5.2.14) it follows that $d(f(x^*), x^*) = 0$. Hence $f(x^*) = x^*$.

The uniqueness of this fixed point can be easily proved as in Theorem 1.

**COROLLARY.** Let $(X,d)$ be a complete metric space and $f : X \to X$ a self mapping. If $\alpha, \gamma, \delta : (0, +\infty) \to (-\infty, +\infty)$ are functions as in Theorem 3 and such that the inequalities (5.2.12), (5.2.13), (5.2.14) and (5.2.15) are satisfied with $\beta = -1$, then $f$ has a unique fixed point.