ABSTRACT. In this paper, we obtain some related fixed point theorems for three metric spaces.

The following fixed point theorem was proved by Nung [1].

THEOREM 1. Let $(X, d)$, $(Y, p)$ and $(Z, \sigma)$ be complete metric spaces and suppose $T$ is a continuous mapping of $X$ into $Y$, $S$ is a continuous mapping of $Y$ into $Z$ and $R$ is a continuous mapping of $Z$ into $X$ satisfying the inequalities

\[
\begin{align*}
&d(RSTx, RSy) \leq \max\{d(x, RSy), d(x, RSTx), p(y, Tx), \sigma(Sy, STx)\}, \\
&p(TRSy, TRz) \leq \max\{p(y, TRz), p(y, TRSy), \sigma(z, Sy), d(Rz, RSy)\}, \\
&\sigma(TRz, STx) \leq \max\{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), p(Tx, TRz)\}
\end{align*}
\]

for all $x$ in $X$, $y$ in $Y$ and $z$ in $Z$, where $0 \leq c < 1$. Then $RST$ has a unique fixed point $u$ in $X$, $TRS$ has a unique fixed point $v$ in $Y$ and $STR$ has a unique fixed point $w$ in $Z$. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

We now prove the following related fixed point theorems:

THEOREM 2. Let $(X, d)$, $(Y, p)$ and $(Z, \sigma)$ be complete metric spaces and suppose $T$ is a mapping of $X$ into $Y$, $S$ is a mapping of $Y$ into $Z$ and $R$ is a mapping of $Z$ into $X$ satisfying the inequalities

\[
\begin{align*}
&d^2(RSy, RSTx) \leq \max\{d(x, RSy)p(y, Tx), p(y, Tx)d(x, RSy)\}, \\
&p^2(TRz, TRSy) \leq \max\{p(y, TRz)\sigma(z, Sy), \sigma(z, Sy)p(y, TRSy), \\
&\quad p(y, TRSy)d(Rz, RSy), d(Rz, RSy)p(y, TRz)\}, \\
&\sigma^2(STx, STRz) \leq \max\{\sigma(z, STx)d(x, Rz), d(x, Rz)\sigma(z, STRz), \\
&\quad \sigma(z, STRz)p(Tx, TRz), p(Tx, TRz)\sigma(z, STx)\}
\end{align*}
\]

for all $x$ in $X$, $y$ in $Y$ and $z$ in $Z$, where $0 \leq c < 1$. If one of the mappings $R, S, T$ is continuous, then $RST$ has a unique fixed point $u$ in $X$, $TRS$ has a unique fixed point $v$ in $Y$ and $STR$ has a unique fixed point $w$ in $Z$. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

PROOF. Let $x_0$ be an arbitrary point in $X$. Define sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in $X$, $Y$ and $Z$ respectively by

\[
x_n = (RST)^nx_0, \quad y_n = Tx_{n-1}, \quad z_n = Sy_n
\]

for $n = 1, 2, \ldots$. 
Applying inequality (2) we have
\[
\rho^2(y_n, y_{n+1}) = \rho^2(TRz_{n-1}, TRSy_n) \\
\leq c \max\{\rho(y_n, y_n)\sigma(z_{n-1}, z_n), \sigma(z_{n-1}, z_n)\sigma(y_n, y_{n+1}), \\
\rho(y_n, y_{n+1})d(x_{n-1}, x_n), d(x_{n-1}, x_n)\} \}
\]
and so
\[
\rho(y_n, y_{n+1}) \leq c \max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}. (4)
\]
Applying inequality (3) we have
\[
\sigma^2(z_n, z_{n+1}) = \sigma^2(STx_{n-1}, STRz_n) \\
\leq c \max\{\sigma(z_n, z_n)d(x_{n-1}, x_n), d(x_{n-1}, x_n)\sigma(z_n, z_{n+1}), \\
\sigma(z_n, z_{n+1})\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1})\sigma(z_n, z_n)\} \\
\]
and so
\[
\sigma(z_n, z_{n+1}) \leq c \max\{d(x_{n-1}, x_n), \rho(y_n, y_{n+1})\} \\
\leq c \max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}. (5)
\]
Applying inequality (1) we have
\[
d^2(x_n, x_{n+1}) = d^2(RSy_n, RSTx_n) \\
\leq c \max\{d(x_n, x_n)\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1})d(x_n, x_{n+1}), \\
d(x_n, x_{n+1})\sigma(z_n, z_{n+1}), \sigma(z_n, z_{n+1})d(x_n, x_n)\} \\
\]
and so
\[
d(x_n, x_{n+1}) \leq c \max\{\rho(y_n, y_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1})\sigma(z_n, z_{n+1})\}. (6)
\]
It now follows easily by induction on using inequalities (4), (5) and (6) that
\[
d(x_n, x_{n+1}) \leq c^n\max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\
\rho(y_n, y_{n+1}) \leq c^n\max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\
\sigma(z_n, z_{n+1}) \leq c^n\max\{d(x_1, x_2), \sigma(z_1, z_2)\}.
\]
Since \(0 \leq c < 1\), it follows that \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) are Cauchy sequences with limits \(u, v\) and \(w\) in \(X, Y\) and \(Z\) respectively.

Now suppose that \(S\) is continuous. Then \(\lim_{n \to \infty} Sy_n = \lim_{n \to \infty} z_n\) and so
\[
Sv = w. (7)
\]
Applying inequality (1) we now have

\[ d^2(RSv, x_{n+1}) = d^2(RSv, RSTx_n) \leq c \max\{d(v, RSv)\rho(v, Tx_n), \rho(v, Tx_n)\rho(v, x_{n+1}), \]

\[ d(x_n, x_{n+1})\sigma(Sv, STx_n), \sigma(Sv, STx_n)\rho(v, RSv)\} \].

Letting \( n \) tend to infinity, it follows on using equation (7) that \( d^2(RSv, u) \leq 0 \) and so

\[ RSv = u. \]  \hfill (8)

Using equation (7), this gives us

\[ Rw = u. \]  \hfill (9)

Using equation (8) and inequality (2) we have

\[ \rho^2(Tu, y_{n+1}) = \rho^2(TRSv, TRSv) \leq c \max\{\rho(y_n, RSTv)\sigma(Sv, Sy_n), \sigma(Sv, Sy_n)\rho(y_n, RSTv), \]

\[ \rho(y_n, RSTv)\rho(y_n, TRSv)\rho(y_n, RSTv)\} \].

Letting \( n \) tend to infinity, it follows on using equation (8) again that \( \rho^2(Tu, v) \leq 0 \) and so

\[ Tu = v. \]  \hfill (10)

It now follows from equations (7), (9) and (10) that

\[ TRSw = TRSw = Tu = v, \]

\[ STRu = STRu = Sv = w, \]

\[ RSTu = RSTu = Rw = u. \]

The same results of course will hold if \( R \) or \( T \) is continuous instead of \( S \).

We now prove the uniqueness of the fixed point \( u \). Suppose that \( RST \) has a second fixed point \( u' \). Then using inequality (1), we have

\[ d^2(u, u') = d^2(RSTu, RSTu') \leq c \max\{d(u', u)\rho(Tu, Tu'), \rho(Tu, Tu')d(u', u'), \]

\[ d(u', u')\sigma(STu, STu'), \sigma(STu, STu')d(u', RSTu)\} = c \max\{d(u, u')\rho(Tu, Tu'), \sigma(STu, STu')d(u, u')\}, \]

which implies that

\[ d(u, u') \leq c \max\{\rho(Tu, Tu')\sigma(STu, STu')\}. \]  \hfill (11)

Further, using inequality (2), we have

\[ \rho^2(Tu, Tu') = \rho^2(TRSTu, TRSTu') \leq c \max\{\rho(Tu', Tu)\sigma(STu, STu'), \sigma(STu, STu')\rho(Tu', Tu'), \]

\[ \rho(Tu', Tu')\rho(Tu', Tu')\} \leq c \max\{\rho(Tu, Tu')\sigma(STu, STu'), d(u, u')\rho(Tu, Tu')\}, \]

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which implies that
\[ \rho(Tu, Tu') \leq c \max\{\sigma(STu, STu'), d(u, u')\}. \]  

(12)

Inequalities (11) and (12) imply that
\[ d(u, u') \leq c\sigma(STu, STu'). \]  

(13)

Finally, using inequality (3), we have
\[
\sigma^2(STu, STu') = \sigma^2(STRSTu, STRSTu') \\
\leq c \max\{\sigma(STu, STu')d(u, u'), d(u, u')\sigma(STu', STu'), \sigma(STu', STu')\rho(Tu, Tu'), \rho(Tu, Tu')\sigma(STu, STu')\} \\
= c \max\{\sigma(STu, STu')d(u, u'), \sigma(STu, STu')\rho(Tu, Tu')\},
\]

which implies that
\[ \sigma(STu, STu') \leq c \max\{d(u, u'), \rho(Tu, Tu')\}. \]  

(14)

It now follows from inequalities (12), (13) and (14) that
\[ d(u, u') \leq c\sigma(STu, STu') \leq c^2\sigma(STu, STu'), \]

and so \( u = u' \), since \( c < 1 \). The fixed point \( u \) of \( RST \) is therefore unique. Similarly, it can be proved that \( v \) is the unique fixed point of \( TRS \) and \( w \) is the unique fixed point of \( STR \). This completes the proof of the theorem.

**Theorem 3.** Let \( (X, d) \), \( (Y, \rho) \) and \( (Z, \sigma) \) be complete metric spaces and suppose \( T \) is a mapping of \( X \) into \( Y \), \( S \) is a mapping of \( Y \) into \( Z \) and \( R \) is a mapping of \( Z \) into \( X \) satisfying the inequalities
\[
d(RSy, RSTx) \max\{d(x, RSy), d(x, RSTx)\} \\
\leq c\sigma(Sy, STx) \max\{\sigma(Sy, STx), d(x, RSTx)\}, \]  

(15)
\[
\rho(TRz, TRSy) \max\{\sigma(y, TRz), \sigma(y, TRSy)\} \\
\leq cd(Rz, RSy) \max\{d(Rz, RSy), \rho(y, TRSy)\}, \]  

(16)
\[
\sigma(STx, STRz) \max\{\sigma(z, STx), \sigma(z, STRz)\} \\
\leq c\sigma(Tx, TRz) \max\{\sigma(Tx, TRz), \sigma(z, STRz)\}, \]  

(17)

for all \( x \) in \( X \), \( y \) in \( Y \) and \( z \) in \( Z \), where \( 0 \leq c < 1 \). If one of the mappings \( R, S, T \) is continuous, then \( RST \) has a unique fixed point \( u \) in \( X \), \( TRS \) has a unique fixed point \( v \) in \( Y \) and \( STR \) has a unique fixed point \( w \) in \( Z \). Further, \( Tu = v, Sv = w \) and \( Rw = u \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \) and define sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( X, Y \) and \( Z \) respectively as in the proof of Theorem 2.

Applying inequality (15) we have
\[
d(x_n, x_{n+1}) \max\{d(x_n, x_n), d(x_n, x_{n+1})\} \\
\leq c\sigma(z_n, z_{n+1}) \max\{\sigma(z_n, z_{n+1}), d(x_n, x_{n+1})\}.
\]
and so either
\[ d^2(x_n, x_{n+1}) \leq c \sigma(z_n, z_{n+1})d(x_n, x_{n+1}) \]
which implies that
\[ d(x_n, x_{n+1}) \leq c \sigma(z_n, z_{n+1}) \]
or
\[ d^2(x_n, x_{n+1}) \leq c \sigma^2(z_n, z_{n+1}) \]
which implies that
\[ d(x_n, x_{n+1}) \leq b \sigma(z_n, z_{n+1}), \]
where \( b = \sqrt{c} \geq c \). Thus either case implies that
\[ d(x_n, x_{n+1}) \leq b \sigma(z_n, z_{n+1}). \tag{18} \]

Applying inequality (17) we have
\[ \sigma(z_n, z_{n+1}) \max \{\sigma(z_n, z_n), \sigma(z_n, z_{n+1})\} \]
\[ \leq c \rho(y_n, y_{n+1}) \max \{\sigma(y_n, y_{n+1}), \sigma(z_n, z_{n+1})\} \]
and it follows as above that
\[ \sigma(z_n, z_{n+1}) \leq b \rho(y_n, y_{n+1}). \tag{19} \]

Applying inequality (16) we have
\[ \rho(y_n, y_{n+1}) \max \{\rho(y_n, y_n), \rho(y_n, y_{n+1})\} \]
\[ \leq c d(x_{n-1}, x_n) \max \{d(x_{n-1}, x_n), \rho(y_n, y_{n+1})\} \]
and it follows as above that
\[ \rho(y_n, y_{n+1}) \leq b d(x_{n-1}, x_n). \tag{20} \]

It now follows from inequalities (18), (19) and (20) that
\[ d(x_n, x_{n+1}) \leq d \sigma(z_n, z_{n+1}) \leq b^2 \rho(y_n, y_{n+1}) \leq \ldots \leq b^n d(x_0, x_1). \]
Since \( 0 \leq b < 1 \), \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are Cauchy sequences with limits \( u, v \) and \( w \) in \( X, Y \) and \( Z \) respectively.

Now suppose that \( S \) is continuous. Then \( \lim_{n \to \infty} S y_n = \lim_{n \to \infty} z_n \) and so
\[ S v = w. \tag{21} \]

Applying inequality (15) we now have
\[ d(RS v, x_n) \max \{d(x_{n-1}, RS v), d(x_{n-1}, x_n)\} \]
\[ \leq c \sigma(S v, z_n) \max \{(S v, z_n), d(x_{n-1}, x_n)\}. \]
Letting \( n \) tend to infinity, it follows on using equation (21) that \( d^2(RS v, u) \leq 0 \) and so
\[ RS v = u. \tag{22} \]
Using equation (21), this gives us

\[ Rw = u. \]  

(23)

Using equation (23) and inequality (16) we have

\[
\rho(Tu, y_{n+1}) \max \{ \rho(y_n, TRw), \rho(y_n, y_{n+1}) \} 
\leq cd(u, x_n) \max \{ d(u, x_n), \rho(y_n, y_{n+1}) \}.
\]

Letting \( n \) tend to infinity, it follows that \( \rho^2(Tu, v) \leq 0 \) and so

\[ Tu = v. \]  

(24)

It now follows from equations (21), (23) and (24) that

\[
TRSv = TRw = Tu = v,
\]

\[
STRw = STu = Sv = w,
\]

\[
RSTu = RSv = Rw = u.
\]

The same results of course will hold if \( R \) or \( T \) is continuous instead of \( S \).

We now prove the uniqueness of the fixed point \( u \). Suppose that \( RST \) has a second fixed point \( u' \). Then using inequality (15), we have

\[
d^2(u, u') = d(RSTu, RSTu') \max \{ d(u, RSTu'), d(u, RSTu) \} 
\leq c\sigma(STu', STu) \max \{ d(STu', STu), d(u, RSTu) \},
\]

which implies that

\[
d(u, u') \leq b\sigma(STu', STu'). \]  

(25)

Further, using inequality (17), we have

\[
s(STu, STu') \max \{ s(STu, STu'), s(STu, STRSTu) \} 
= s(STRSTu, STu) \max s(STu, STu') s(STu, STRSTu) 
\leq cp(Tu', TRSTu) \max \{ \rho(Tu', TRSTu), \sigma(STu, STRSTu) \},
\]

which implies that

\[
s(STu, STu') \leq bp(Tu, Tu'). \]  

(26)

Finally, using inequality (16), we have

\[
\rho(Tu, Tu') \max \{ \rho(Tu, TRSTu'), \rho(Tu, TRSTu) \} 
\leq cd(RSTu', RSTu) \max \{ d(RSTu', RSTu), \rho(Tu, TRSTu) \},
\]

which implies that

\[
\rho(Tu, Tu') \leq bd(u, u'). \]  

(27)

Since \( b < 1 \), it now follows immediately from inequalities (25), (26) and (27) that \( u = u' \). The fixed point \( u \) of \( RST \) is therefore unique. Similarly, it can be proved that \( v \) is the unique fixed point of \( TRS \) and \( w \) is the unique fixed point of \( STR \).

This completes the proof of the theorem.