4. MAP/PH/1 QUEUE WITH WORKING VACATIONS, VACATION INTERRUPTIONS AND N POLICY

In chapters 2 and 3, we considered the case of a two server system where the second server goes on a vacation, whenever no customer is found waiting at the end of a service. This server followed a simple vacation policy in the model discussed in chapter 2 and a working vacation policy in the model of chapter 3. These two queueing models were with Poisson arrivals and exponential service times. In reality these assumptions are very restrictive though they make the system analytically more tractable. The traffic in modern communication network is highly irregular. Of late to model systems with repeated calls and bursty arrivals MAP (Markovian arrival process) is used. The MAP is a tractable class of point process which is in general nonrenewal. However by choosing the parameters of the MAP appropriately the underlying arrival process can be made a renewal process. The MAP can represent a variety of processes which includes, as special cases, the Poisson process, the phase-type renewal processes, the Markov modulated Poisson process and superpositions of these.

Here we consider a single server queueing model in which customers arrive according to a Markovian arrival process with representation \((D_0, D_1)\) of

\(^0\) To appear in Applied Mathematical Modelling
order $m$. The service times are assumed to be of phase type with representation $(\alpha, T)$ of order $n$. At a service completion epoch the server, finding the system empty, takes a vacation. The duration of vacation is assumed to be exponentially distributed with parameter $\eta$. A customer arriving during a vacation will be served at a lower rate. To be precise, the service time during vacation follows phase type distribution with representation $(\alpha, \theta T)$, $0 < \theta < 1$. Thus $\mu = [\alpha(-T)^{-1}e]^{-1}$ is the normal service rate and $\theta \mu$ is the rate of the vacation mode of service. The server continues to serve at this rate until either the vacation clock expires or the queue length hits the threshold value $N$, $1 \leq N < \infty$. When either of these two occurs the server instantaneously switches over to the normal rate and continues to serve at this rate until the system becomes empty.

Let $Q^* = D_0 + D_1$ be the generator matrix of the arrival process and $\pi$ be the stationary probability vector of the Markov process with generator $Q^*$. That is, $\pi$ is the unique (positive) probability vector satisfying

$$\pi Q^* = 0, \quad \pi e = 1. \quad (4.1)$$

The constant $\lambda = \pi D_1 e$, referred to as the *fundamental rate*, gives the expected number of arrivals per unit of time in the stationary version of the MAP. Often, in model comparisons, it is convenient to select the time scale of the MAP so that $\lambda$ has a certain value. That is accomplished, in the continuous MAP case, by multiplying the coefficient matrices $D_0$ and $D_1$, by the appropriate common constant.
4.1 The QBD process

The model described in Section 1 can be studied as a quasi-birth-and-death (QBD) process. First, we set up necessary notations.

Define \( N(t) \) to be the number of customers in the system at time \( t \),

\[
S_1(t) = \begin{cases} 
0, & \text{if the service is in vacation mode}, \\
1, & \text{if the service is normal},
\end{cases}
\]

\( S_2(t) \) is the phase of the service process when the server is busy and \( M(t) \) to be the phase of the arrival process at time \( t \). It is easy to verify that \( \{(N(t), S_1(t), S_2(t), M(t)) : t \geq 0\} \) is a level independent quasi-birth-and-death process (LIQBD) with state space

\[
\Omega = \bigcup_{i=0}^{\infty} l(i)
\]

where

\[
l(0) = \{(0,1), (0,2), \ldots, (0,m)\}
\]

and for \( i \geq 1 \),

\[
l(i) = \{(i,j_1,j_2,k) : j_1 = 0 \text{ or } 1; 1 \leq j_2 \leq n; 1 \leq k \leq m\}.
\]

Note that when \( N(t) = 0 \), server will be on vacation and so \( S_1(t) \) and \( S_2(t) \) do not play any role and will not be tracked. The only other component in the state vector would be \( M(t) \).
The generator, $Q$, of the $QBD$ process under consideration is of the form

$$Q = \begin{pmatrix} D_0 & C_0 \\ C_2 & B_1 & I \otimes D_1 \\ & \ddots & \ddots & \ddots \\ & & B_2 & B_1 & I \otimes D_1 \\ & & & B_2 & B_1 & e \otimes I \otimes D_1 \\ & & & & \epsilon_2(2) \otimes T^0 \alpha \otimes I & A_1 & A_0 \\ & & & & & A_2 & A_1 & A_0 \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where the (block) matrices appearing in $Q$ are as follows.

$$C_0 = [\alpha \otimes D_1 \ 0], \quad C_2 = \begin{bmatrix} \theta T^0 \otimes I \\ T^0 \otimes I \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \theta T \oplus D_0 - \eta I & \eta I \\ O & T \oplus D_0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \theta T^0 \alpha \otimes I & O \\ O & T^0 \alpha \otimes I \end{bmatrix},$$

$$A_0 = I \otimes D_1, \quad A_1 = T \oplus D_0, \quad A_2 = T^0 \alpha \otimes I.$$
4.1. The \textit{QBD} process

4.1.1 The steady-state probability vector

Defining $A = A_0 + A_1 + A_2$ and $\delta$ to be the steady-state probability vector of the irreducible matrix $A$, it is easy to verify that the vector $\delta$ satisfying

$$\delta A = 0, \quad \delta e = 1,$$

is given by

$$\delta = (\mu \alpha (-T)^{-1} \otimes \pi), \quad (4.2)$$

where $\pi$ as given in (4.1).

The condition $\delta A_0 e < \delta A_2 e$, required for the stability of the queueing model under study (see [50]) reduces to $\lambda < \mu$.

Let $x$ be the steady-state probability vector of $Q$. Partitioning this vector as

$$x = (x_0, x_1, x_2, \ldots, x_N, x_{N+1}, \ldots),$$

where $x_0$ is of dimension $m$; $x_1, x_2, \ldots x_N$ are of dimension $2mn$; and $x_{N+1}$, $x_{N+2}, \ldots$ are of dimension $mn$. Under the condition that $\lambda < \mu$, the steady-state probability vector $x$ is obtained as follows.

$$x_{N+i} = x_{N+1} R^{i-1}, \quad i \geq 1, \quad (4.3)$$

where the matrix $R$ is the minimal nonnegative solution to the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0. \quad (4.4)$$
and the vectors $x_0, \cdots, x_{N+1}$ are obtained by solving

$$x_0D_0 + x_1C_2 = 0,$$

$$x_0C_0 + x_1B_1 + x_2B_2 = 0,$$

$$x_{i-1}(I \otimes D_1) + x_iB_1 + x_{i+1}B_2 = 0, \quad 2 \leq i \leq N - 1,$$

$$x_{N-1}(I \otimes D_1) + x_NB_1 + x_{N+1}(e_2'(2) \otimes T^0 \alpha \otimes I) = 0,$$

$$x_N(e \otimes I \otimes D_1) + x_{N+1}(A_1 + RA_2) = 0,$$

subject to the normalizing condition

$$\sum_{i=0}^{N} x_i e + x_{N+1}(I - R)^{-1} e = 1.$$ 

The computation of the vectors $x_0, \cdots, x_{N+1}$ can be carried out by exploiting the special structure of the coefficient matrices and the details are omitted. For use in the sequel, we partition $x_i = (u_i, v_i), 1 \leq i \leq N$, where $u_i$ and $v_i$ are of dimension $mn$.

4.1.2 The stationary waiting time distribution in the Queue

The stationary waiting time distribution in the queue of a customer is derived here. We obtain this by conditioning on the fact that at an arrival epoch the server is serving in normal mode or in vacation mode. First note that an arriving customer will enter into service immediately (at a lower service
rate) when the server is on vacation. Otherwise, the customer has to wait before getting into service (either at a lower rate or normal rate).

4.1.3 Conditional waiting time in the queue (Normal mode)

Here we condition that an arriving customer finds the server busy serving in normal mode. First note that in this case, the waiting time is always positive. We now define $z_{i,j}$ to be the steady-state probability that an arrival finds the server busy in normal mode with the current service in phase $j$, and the number of customers in the system including the current arrival to be $i$, for $1 \leq j \leq n, i \geq 2$. Let $z_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,n})$ and $z = (0, z_2, z_3, \ldots)$. Then it is easy to verify that

$$z_i = \begin{cases} 
    v_i (I \otimes \frac{D_i}{\lambda} e), & 2 \leq i \leq N, \\
    (u_N + v_N)(I \otimes \frac{D_i}{\lambda} e), & i = N + 1, \\
    x_{i-1}(I \otimes \frac{D_i}{\lambda} e), & i \geq N + 2.
\end{cases}$$

The waiting time may be viewed as the time until absorption in a Markov chain with a highly sparse structure. The state space (that includes the arriving customer in its count) of this Markov chain is given by $\Omega_1 = \{*, \ldots\} \cup \{(i,j) : i \geq 2, 1 \leq j \leq n\}$. The state $*$ corresponds to the absorbing state indicating the completion of waiting for the service. It is easy to verify that
the generator, $\tilde{Q}_1$, of this Markov process is of the form

$$
\tilde{Q}_1 = \begin{pmatrix}
0 & O \\
T^\alpha & T \\
T^\alpha & T \\
\ddots & \ddots
\end{pmatrix}
$$

Define $W(t), t > 0$ to be the probability that an arriving customer will enter into service no later than time $t$ conditioned on the fact that the service is in normal mode. Let $\tilde{W}_{normal}(s)$ denote the Laplace-Stieltjes transform of the conditional stationary waiting time in the queue of an arriving customer during the normal service mode. Using the structure of $\tilde{Q}_1$ it can readily be verified that the following result holds good.

**Theorem 4.1.1.** The LST of the conditional waiting time distribution of an arriving customer, finding the server busy in normal mode, is given by

$$
\tilde{W}_{normal}(s) = c \sum_{i=2}^{\infty} z_i (sI - T)^{-1} T^0 \alpha (sI - T)^{-1} T^0 [sI - T]^{-2}, \quad Re(s) \geq 0, \quad (4.5)
$$

where the normalizing constant $c$ is given by

$$
c = \left[ \sum_{i=2}^{\infty} z_i e \right]^{-1}. \quad (4.6)
$$

**Note:** The conditional mean waiting time, $\mu'_{normal}$ in the queue of an arrival finding the server to be busy in normal mode soon after the arrival is
4.1. The $QBD$ process

calculated as

$$
\mu'_\text{normal} = -\hat{W}'_\text{normal}(0) = c \sum_{i=2}^{\infty} \tilde{z}_i (-T)^{-1} e + \frac{c}{\mu} \sum_{i=2}^{\infty} (i-2) z_i e.
$$

Substituting for $z_i$ in the last equation, we get $\mu'_\text{normal}$ in the simplified form as

$$
\mu'_\text{normal} = \frac{c}{\lambda} \left[ \sum_{i=1}^{N} v_i + u_N + x_{N+1}(I-R)^{-1} \right] (-T)^{-1} e \otimes D_1 e] \\
+ \frac{c}{\lambda} \sum_{i=1}^{\infty} v_i + (N-1) u_N + N x_{N+1}(I-R)^{-1} + x_{N+1} R(I-R)^{-2} \right] e \otimes D_1 e].
$$

4.1.4 Conditional waiting time in the queue (vacation mode)

The conditional stationary waiting time in the queue of an arriving customer given that the server is busy in vacation mode at that instant is derived here. Let $w_{i,j_2,k}; 1 \leq i \leq N; 1 \leq j_2 \leq n; 1 \leq k \leq m$ denote the steady-state probability that a customer immediately after arrival, finds the server busy in vacation mode with the service in phase $j_2$ and the number of customers in the system (including the current arrival) to be $i$ and the arrival process is in phase $k$. Let $w_i = (w_{i,1,1}, \ldots, w_{i,n,m})$. It is easy to verify that

$$
w_i = \begin{cases} \\
\quad x_0 (\alpha \otimes \frac{D_k}{\lambda}), & i = 1, \\
\quad u_{i-1} (I \otimes \frac{D_k}{\lambda}), & 2 \leq i \leq N.
\end{cases}
$$

Observe that the conditional waiting time in the queue of an arriving customer, finding the server busy in vacation mode, depends on the future
arrivals due to threshold $N$ placed on the system for bringing the service rate to normal. Also note that with probability $d \nu_1 e$ (the normalizing constant $d$ is given below) an arriving customer will enter into service immediately with service in vacation mode. Thus, for the case of positive waiting time in the queue for an arriving customer, we need to keep track of the phase of the arrival process until the service rate comes to normal mode either due to meeting the threshold $N$ or due to the vacation getting completed. Towards this end, we define the following set of states.

Let $(i, j, j_2, k) : 1 \leq i \leq N - 1; 1 \leq j \leq i; 1 \leq j_2 \leq n; 1 \leq k \leq m,$ denote the state that corresponds to the server being in vacation mode with $i$ customers in the queue; the arriving customer’s position in the queue is $j$; the current service is in phase $j_2$ and the arrival process is in phase $k$.

Define $(i^*, j_2) : 1 \leq i^* \leq N - 1; 1 \leq j_2 \leq n,$ to be the state that corresponds to the server serving in normal mode with the position of the tagged customer in the queue being $i^*$ and the current service in phase $j_2$.

Let $i = \{(i, j, j_2, k) : 1 \leq j \leq i; 1 \leq j_2 \leq n; 1 \leq k \leq m\}, 1 \leq i \leq N - 1,$ and

$i^* = \{(i^*, j_2) : 1 \leq j_2 \leq n\}, 1 \leq i^* \leq N - 1.$

Before we formally state the result we need the following notations.

- $\tilde{I}_r$ is a matrix of dimension $r \times r + 1$ of the form

$$\tilde{I}_r = \begin{pmatrix} I_r & O \end{pmatrix}, \quad 1 \leq r \leq N - 2.$$  

- $\hat{I}_r$ is a matrix of dimension $r \times N - 1$ of the form

$$\hat{I}_r = \begin{pmatrix} I_r & O \end{pmatrix}, \quad 1 \leq r \leq N - 1.$$  


4.1. The QBD process

- $\tilde{I}_r$ is a matrix of dimension $r \times r - 1$ of the form

$$\tilde{I}_r = \begin{pmatrix} O \\ I_{r-1} \end{pmatrix}, \quad 2 \leq r \leq N - 1.$$ 

- $I_r$ is the identity matrix of dimension $r$

- $d$ is the normalizing constant given by $d = \left[ \sum_{i=1}^{N} w_i e \right]^{-1}$.

Let

$$L_{1,1} = \begin{pmatrix} T & T^0\alpha & T \\ T^0\alpha & T & T \\ \cdots & \cdots & \cdots \end{pmatrix}, \quad L_{2,1} = \begin{pmatrix} \eta\tilde{I}_1 \otimes I \otimes e \\ \eta\tilde{I}_2 \otimes I \otimes e \\ \vdots \\ \eta\tilde{I}_{N-2} \otimes I \otimes e \\ I_{N-1} \otimes (\eta I \otimes e + I \otimes D_1 e) \end{pmatrix},$$ 

$$L_{2,2} = \begin{pmatrix} \tilde{B}_1 \quad \tilde{I}_1 \otimes I \otimes D_1 \\ F_2 \quad \tilde{I}_2 \otimes \tilde{B}_1 \quad \tilde{I}_2 \otimes I \otimes D_1 \\ F_3 \quad I_3 \otimes \tilde{B}_1 \quad \tilde{I}_3 \otimes I \otimes D_1 \\ \cdots & \cdots & \cdots \\ F_{N-1} \quad I_{N-1} \otimes \tilde{B}_1 \end{pmatrix},$$

and

$$\tilde{B}_1 = (\theta T \oplus D_0) - \eta I; \quad F_K = \theta \tilde{I}_K \otimes T^0\alpha \otimes I, \quad 2 \leq K \leq N - 1. \quad (4.7)$$

Under this setup, it can readily be verified that
Theorem 4.1.2. The conditional waiting time distribution in the queue of a customer, finding the server in vacation mode on arrival, is of phase type with representation $(\gamma, L)$ of order $[(N - 1)n + \frac{1}{2}N(N - 1)mn]$ where

$$\gamma = d(0, w_2, e'_2(2) \otimes w_3, e'_3(3) \otimes w_4, \ldots, e'_{N-1}(N - 1) \otimes w_N),$$

and

$$L = \begin{pmatrix} L_{1,1} & 0 \\ L_{2,1} & L_{2,2} \end{pmatrix}.$$ 

Note: The conditional mean waiting time, $\mu'_{\text{vacation}}$, in the queue of an arrival finding the server busy in vacation mode on arrival is calculated as $\mu'_{\text{vacation}} = \gamma(-L)^{-1}e$. The computation of this mean is achieved by exploiting the special structure of $\gamma$ and $L$. We will briefly present the steps involved in this.

Define

$$\gamma(-L)^{-1} = (a, b),$$

and partition the vectors $a$ and $b$ as

$$a = (a_1, \ldots, a_{N-1}),$$

$$b = (b_{1,1}, b_{2,1}, b_{2,2}, \ldots, b_{N-1,1}, \ldots, b_{N-1,N-1}),$$

where $a_i, 1 \leq i \leq N - 1$, is of dimension $n$ and $b_{i,j}, 1 \leq j \leq i, 1 \leq i \leq N - 1$, is of dimension of $mn$. The vectors $a$ and $b_{i,j}$ are ideally suited for solving using any of the well-known methods such as (block) Gauss-Seidel. The
4.1. The QBD process

The necessary equations are as follows.

\[ a_i = a_2 T^0 \alpha (-T)^{-1} + \eta \sum_{r=1}^{N-1} b_{r,1} (-T^{-1} \otimes e) + b_{N-1,1} (-T^{-1} \otimes D_1 e), \]

\[ a_i = a_{i+1} T^0 \alpha (-T)^{-1} + \eta \sum_{r=1}^{N-1} b_{r,i} (-T^{-1} \otimes e) + b_{N-1,i} (-T^{-1} \otimes D_1 e), \quad 2 \leq i \leq N-2, \]

\[ a_{N-1} = \eta b_{N-1,N-1} (-T^{-1} \otimes e) + b_{N-1,N-1} (-T^{-1} \otimes D_1 e), \]

\[ b_{i,1} = [w_2 + \theta b_{2,2} (T^0 \alpha \otimes I)] (-\hat{B}_1)^{-1}, \]

\[ b_{i,1} = [b_{i-1,1} (I \otimes D_1) + \theta b_{i+1,2} (T^0 \alpha \otimes I)] (-\hat{B}_1)^{-1}, \quad 2 \leq i \leq N-2, \]

\[ b_{i,j} = [b_{i-1,j} (I \otimes D_1) + \theta b_{i+1,j+1} (T^0 \alpha \otimes I)] (-\hat{B}_1)^{-1}, \quad 2 \leq j \leq i-1; \quad 2 \leq i \leq N-2, \]

\[ b_{i,i} = [w_{i+1} + \theta b_{i+1,i+1} (T^0 \alpha \otimes I)] (-\hat{B}_1)^{-1}, \quad 2 \leq i \leq N-2, \]

\[ b_{N-1,j} = b_{N-2,j} (I \otimes D_1) (-\hat{B}_1)^{-1}, \quad 1 \leq j \leq N-2, \]

\[ b_{N-1,N-1} = w_N (-\hat{B}_1)^{-1}, \]

subject to the condition

\[ a_1 T^0 + \theta \sum_{i=1}^{N-1} b_{i,1} (T^0 \otimes e) = 1 - dw_1 e. \]
4. MAP/PH/1 Queue with working vacations, vacation interruptions and N Policy

Once $a_i, 1 \leq i \leq N - 1$, and $b_{i,j}, 1 \leq j \leq i; 1 \leq i \leq N - 1$, are extracted from the above equations, the mean $\mu_{\text{vacation}}'$ is given by

$$\mu_{\text{vacation}}' = \sum_{i=1}^{N-1} \left[ a_i e + \sum_{j=1}^i b_{i,j} e \right].$$

The stationary waiting time in the queue

From the knowledge of conditional stationary waiting time in the queue, one can get the (unconditional) stationary waiting time in the queue; the details are omitted.

Note: The (unconditional) mean, $\mu_{\text{WTQ}}'$, waiting time of a customer in the queue is obtained as

$$\mu_{\text{WTQ}}' = \frac{1}{\lambda} \left[ \sum_{i=1}^N v_i + u_N + x_{N+1}(I - R)^{-1} \right] \left[ (T)^{-1} e \otimes D_1 e \right]$$

$$+ \frac{1}{\lambda \mu} \left[ \sum_{i=1}^\infty v_i + (N - 1) u_N + N x_{N+1}(I - R)^{-1} + x_{N+1}R(I - R)^{-2} \right] \left[ e \otimes D_1 e \right]$$

$$+ \frac{1}{d} \sum_{i=1}^{N-1} \left[ a_i e + \sum_{j=1}^i b_{i,j} e \right].$$

4.2 Analysis of slow service mode

In this section we will discuss the duration of the server spending in slow service mode as well as the number of visits to level 0 before hitting normal service mode.
4.2. Analysis of slow service mode

4.2.1 Distribution of a slow service mode

The duration, $T_{\text{slow}}$, in slow service mode is defined as the time the server starts in slow service mode (through initiating a working vacation) until either the server takes another vacation or the server gets back to normal mode through the working vacation expiring or the working vacation is interrupted as the queue length hits the threshold value $N$. In this section we will show that the random variable $T_{\text{slow}}$ can be studied as the time until absorption in a finite state continuous time Markov chain with two absorbing states. We first define

$$\gamma_M = c_1 (\alpha \otimes x_0 D_1, 0),$$

$$M = \begin{pmatrix} \tilde{B}_1 & I \otimes D_1 \\ \theta(T^0 \alpha \otimes I) & \tilde{B}_1 & I \otimes D_1 \\ \theta(T^0 \alpha \otimes I) & \tilde{B}_1 & I \otimes D_1 \\ \vdots & \vdots & \vdots \\ \theta(T^0 \alpha \otimes I) & \tilde{B}_1 \end{pmatrix},$$

$$M^0_1 = \begin{pmatrix} \theta(T^0 \otimes e) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad M^0_2 = \begin{pmatrix} \eta e \\ \eta e \\ \vdots \\ \eta e + (e \otimes D_1 e) \end{pmatrix},$$

where $c_1 = [x_0 D_1 e]^{-1}$ is the normalizing constant and $\tilde{B}_1$ is as given in (4.7).

The matrix $M$ is of dimension $Nmn$. First note that the probability, $p_{\text{slow}}$, that the server will serve only in slow mode before taking another vacation
4. $MAP/PH/1$ Queue with working vacations, vacation interruptions and $N$ Policy

is given by $p_{\text{slow}} = \gamma_M (M)^{-1} M_1^d$. We now have the following result.

**Theorem 4.2.1.** The (conditional) probability density function of $T_{\text{slow}}$, conditioned on the fact that the slow service mode ends through the server taking another vacation, is given by

$$f_{T_{\text{slow}}}(y) = \frac{1}{p_{\text{slow}}} \gamma_M e^{My} M_1^d, \quad y \geq 0. \quad (4.8)$$

Given that the slow service mode ends through the server taking another vacation the (conditional) mean time spent in slow mode can be calculated as

$$\mu'_{SM} = \frac{1}{p_{\text{slow}}} \gamma_M (M)^{-2} M_1^d. \quad (4.9)$$

**Note:** 1. The special structure of $\gamma_M, M,$ and $M_1^d$ is to be exploited when computing this mean. The details are similar to the computation of $\mu'_{\text{vacation}}$ and hence omitted.

2. By a similar argument we can get the (conditional) probability density function of $T_{\text{slow}}$ and the mean, conditioned on the fact that the server ends the slow service mode by entering into the normal rate. The details are omitted.

### 4.2.2 Distribution of the number of visits to level 0 before hitting normal service mode

We consider the queueing system at an arrival epoch that finds the server in vacation. At this instant the service will start in slow mode. The quantity that is of interest here is the probability mass function $\{p_k, k \geq 0\}$, of the
number of visits to level $0$ before hitting normal service mode. This mass function and its associated measures such as mean and standard deviation, play an important role in the qualitative study of the model under consideration. Using the set up in 4.2.1 it can easily be verified that

$$p_k = \gamma_M (-M)^{-1} B^k M_2^0, \ k \geq 0, \quad (4.10)$$

where

$$B = \theta \left[ (e_N(1) e_N'(1) \otimes T^0 \alpha \otimes (-D_0)^{-1} D_1) \right] (-M)^{-1}. \quad (4.11)$$

**Note:** It is easy to see that the mean number of visits to level $0$ before hitting level $N + 1$, $\mu_{NVZ}$, is obtained as

$$\mu_{NVZ} = \gamma_M (-M)^{-1} B (I - B)^{-2} M_2^0. \quad (4.12)$$

The computation of $\mu_{NVZ}$ can be carried out by exploiting the special structure of $\gamma_M, M,$ and $B$. Below, we will outline the main steps. Towards this end, we first define

$$\gamma_M (-M)^{-1} = (d_1, \cdots, d_N), \quad (4.13)$$

where the vectors $d_i, 1 \leq i \leq N,$ are of dimension $nm$, and their computation is very similar to the one discussed in finding $\mu'_{\text{vacation}}$. From (4.11) it is clear
that $B$ is of the form

$$B = \begin{pmatrix} B_1 & B_2 & B_N \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

where the matrices $B_i, 1 \leq i \leq N$, of order $nm$ are obtained by solving the following equations that are ideally suited for any of the well-known methods such as (block) Gauss-Seidel.

$$B_1 = \theta[B_2(T^0 \alpha \otimes I) + (T^0 \alpha \otimes (D_0)^{-1} D_1)](-\tilde{B}_1)^{-1},$$

$$B_i = [B_{i-1}(I \otimes D_1) + \theta B_{i+1}(T^0 \alpha \otimes I)](-\tilde{B}_1)^{-1}, \quad 2 \leq i \leq N - 1,$$

$$B_N = B_{N-1}(I \otimes D_1)(-\tilde{B}_1)^{-1},$$

subject to the condition

$$\theta B_1(T^0 \otimes e) + B_N(e \otimes D_1 e) + \eta \sum_{i=1}^{N} B_i e = \theta(T^0 \otimes e),$$

and $\tilde{B}_1$ is as given in (4.7). Using the facts that

$$p_{\text{slow}} = \theta d_1(T^0 \otimes e) \quad \text{and} \quad \mu_{NVZ} = \gamma_M(-M)^{-1}(I - B)^{-2}M_2^0 - 1,$$
and the special form of $B$, it can easily be verified that

$$\mu_{NVZ} = \theta d_1 (I - B_1)^{-1} (T^0 \otimes e).$$

4.2.3 The uninterrupted duration of a vacation

The duration of the time the server is in uninterrupted vacation(s) is the interval between the epoch at which the server goes on vacation and the next arrival epoch. It is easy to verify that this duration is of phase type with representation $(\xi, D_0)$ of dimension $m$, where $\xi = c_2 (\theta u_1 + v_1) (T^0 \otimes I)$ and $c_2$ is the normalizing constant given by $c_2 = [(\theta u_1 + v_1) (T^0 \otimes e)]^{-1}$. The mean, $\mu_{UIV}$, is calculated as $\mu_{UIV} = \xi (-D_0)^{-1} e$.

4.2.4 Key system performance measures

In this section we list a number of key system performance measures to bring out the qualitative aspects of the model under study. The measures are listed below along with their formulae for computation.

1. Probability that the server is on vacation: $P_{VAC} = x_0 e$.

2. Probability that the server is serving at a lower rate: $P_{LR} = \sum_{i=1}^{N} u_i e$.

3. Probability that the server is serving at a normal rate rate:

$$P_{NR} = \sum_{i=1}^{N} v_i e + x_{N+1} (I - R)^{-1} e.$$

4. Mean number of customers in the system:

$$\mu_{NS} = \sum_{i=1}^{N} i (u_i + v_i) e + N x_{N+1} (I - R)^{-1} e + x_{N+1} (I - R)^{-2} e.$$
4.3 Numerical Results

For the arrival process we consider the following five sets of matrices for $D_0$ and $D_1$.

1. Erlang ($ERA$)
   \[
   D_0 = \begin{pmatrix}
   -5 & 5 \\
   -5 & 5 \\
   -5 & 5 \\
   -5 & 5 \\
   -5 & 5
   \end{pmatrix},
   \quad
   D_1 = \begin{pmatrix}
   \end{pmatrix}
   \]

2. Exponential ($EXA$)
   \[
   D_0 = (-1),
   D_1 = (1)
   \]

3. Hyperexponential ($HEA$)
   \[
   D_0 = \begin{pmatrix}
   -10 & 0 \\
   0 & -1
   \end{pmatrix},
   D_1 = \begin{pmatrix}
   9 & 1 \\
   0.9 & 0.1
   \end{pmatrix}
   \]

4. MAP with negative correlation ($MNA$)
   \[
   D_0 = \begin{pmatrix}
   -2 & 2 & 0 \\
   0 & -2 & 0 \\
   0 & 0 & -450.5
   \end{pmatrix},
   D_1 = \begin{pmatrix}
   0 & 0 & 0 \\
   0.02 & 0 & 1.98 \\
   445.995 & 0 & 4.505
   \end{pmatrix}
   \]

5. MAP with positive correlation ($MPA$)
   \[
   D_0 = \begin{pmatrix}
   -2 & -2 & 0 \\
   0 & -2 & 0 \\
   0 & 0 & -450.5
   \end{pmatrix},
   D_1 = \begin{pmatrix}
   0 & 0 & 0 \\
   1.98 & 0 & 0.02 \\
   4.505 & 0 & 445.995
   \end{pmatrix}
   \]

All these five $MAP$ processes are normalized so as to have an arrival rate of 1. However, these are qualitatively different in that they have different variance...
4.3. Numerical Results

and correlation structure. The first three arrival processes, namely ERA, EXA, and HEA, correspond to renewal processes and so the correlation is 0. The arrival process labelled MNA has correlated arrivals with correlation between two successive inter-arrival times given by -0.4889 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.4889. The ratio of the standard deviations of the inter-arrival times of these five arrival processes with respect to ERA are, respectively, 1, 2.2361, 5.0194, 3.1518, and 3.1518.

For the service time distribution we consider the following two phase type distributions.

1. Erlang (ERS)
   \[ \alpha = (1, 0) \ T = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix} \]

2. Hyperexponential (HES)
   \[ \alpha = (0.9, 0.1) \ T = \begin{pmatrix} -1.90 & 0 \\ 0 & -0.19 \end{pmatrix} \]

The above two distributions will be normalized to have a specific mean in our illustrative example. Note that these are qualitatively different in that they have different variances. The ratio of the standard deviation of HES to that of ERS is 3.1745.

**ILLUSTRATIVE EXAMPLE 4.1:** The purpose of this example is to see how various system performance measures behave under different scenarios. We fix \( \lambda = 1, \mu = 1.1, \) and \( \theta = 0.6. \) First we look at the effect of varying
$N$ and $\eta$ on the performance measures: (conditional) mean duration of service in slow mode which ends in the server taking another vacation and the mean number of visits to level zero before hitting the normal service mode. In the following we summarize the observations based on the graphs of these performance measures.

- Consider figures 4.1 and 4.2. An increase in $\eta$ leads to a decrease in the mean duration of vacation. Hence a switching from the lower service rate to the normal one occurs more frequently. Once the service rate is brought back to normal, the server clears out the customers at a faster rate. So the measure $P_{VAC}$ appears to increase as $\eta$ increases. This is true for all values of $N$ and for all combinations of arrival and service processes under study. As $N$ increases the duration of vacation mode of service gets extended, as is expected. Due to the slow service rate the customers get accumulated faster. So $P_{VAC}$ decreases until the service rate gets to normal. Also note that the probability, $P_{LR}$, that the server is serving at a low rate increases as $N$ is increased (for fixed $\eta$) for all combinations of arrival and service distributions. This in turn will cause the probability, $P_{NR}$, of the server serving under normal mode to decrease as $N$ increases. As expected, the measure $P_{NR}$ appears to increase with increasing $\eta$. When comparing the mean duration of service in slow mode, we notice (for fixed $N$ and $\eta$) that HES yield a lower value as opposed to ERS. This is the case for all five arrival processes considered.
4.3. Numerical Results

\[ \lambda = 1, \mu = 1.1, \theta = 0.6 \]

Fig. 4.1: Mean duration in slow mode - Erlang services
4. MAP/PH/1 Queue with working vacations, vacation interruptions and $N$ Policy

$\lambda = 1, \mu = 1.1, \theta = 0.6$

**Fig. 4.2:** Mean duration in slow mode - hyperexponential services
4.3. Numerical Results

\[ \lambda = 1, \mu = 1.1, \theta = 0.6 \]

Fig. 4.3: Mean number of visits to level zero - Erlang services
4. MAP/PH/1 Queue with working vacations, vacation interruptions and

\[ N \] Policy

\[ \lambda = 1, \mu = 1.1, \theta = 0.6 \]

Fig. 4.4: Mean number of visits to level zero - hyperexponential services
4.3. Numerical Results

- Referring to Figures 4.3 and 4.4, we note that as \( \eta \) increases, the measure \( \mu_{NVZ} \) appears to decrease in all cases, as expected, for any fixed \( N \). Among renewal arrivals, those with larger variation yields a smaller value for this measure. That is, \( HEA \) has a smaller value compared to \( EXA \) and \( EXA \) has a smaller value compared to \( ERA \). Among correlated arrivals, \( MPA \) has a higher value than \( MNA \). It is worth pointing out that both \( MNA \) and \( MPA \) processes have the same mean and variance, but \( MPA \) has a positive correlation while \( MNA \) has a negative correlation. This indicates the significant role played by correlation. As \( N \) increases, this measure appears to increase monotonically to a limiting value (which depends on \( \eta \) as well as on the arrival and service time distributions). It should be noted that the rate of approach is higher for larger values of \( \eta \). That is, the impact of \( N \) on this measure decreases as \( \eta \) increases. We notice that this measure appears to have a larger value when services are changed from Erlang to hyperexponential. When comparing this measure for various distributions (for fixed \( N \) and \( \eta \)), we notice that \( HES \) yield a higher value as opposed to \( ERS \). This is the case for all five arrival processes considered.

Now we look at the unconditional mean waiting time, \( \mu'_{WTQ} \), in the queue of a customer. The values of this measure as functions of \( N \) and \( \eta \) under different scenarios are displayed in Table 4.1. Some key observations are as follows.

- As is to be expected, the mean is a non-increasing function of \( \eta \) (for fixed \( N \)) and is a non-decreasing function of \( N \) (for fixed \( \eta \)). This is
the case for all combinations of arrival and service processes. However, the rate of change is much smaller in the case of \textit{MPA} as compared to the other arrivals.

- The mean is significantly larger for \textit{MPA} case indicating the role played by the (positively) correlated arrivals.

- For all arrivals except \textit{MPA} arrivals, we notice the mean changes significantly as a function of $\eta$ when $N$ becomes large. This is due to the fact that for large $N$ the mean waiting time can only be reduced through an increase in $\eta$ (which will decrease the duration of the slow service period).
4.3. Numerical Results

The unconditional mean waiting time in the queue ($\mu_{WTQ}'$)

Table 4.1

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<tr>
<th>N</th>
<th>$\eta$</th>
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<th>$\text{HEA}$</th>
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<th>$\text{MPA}$</th>
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