Chapter 2

The zero-divisor graph of a meet-semilattice

2.1 Introduction

The study of algebraic structures, using properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers assigning a graph to a ring, semigroup and investigation of algebraic properties of the ring, semigroup using the associated graph.

The concept of a zero-divisor graph of a commutative ring was introduced by I. Beck in [8], he defined and studied coloring of rings. Anderson and Livingston in [6] studied graphs on commutative rings. Let $R$ be a commutative ring with 1 and let $Z(R)$ be its set of all zero-divisors. They associated a (simple) graph $\Gamma(R)$ to $R$ with vertex set $Z(R)^* = Z(R) - \{0\}$, the set of all nonzero zero-divisors of $R$ and distinct $x, y \in Z(R)^*$ are adjacent if and only if $xy = 0$ and called this
graph as the zero-divisor graph of $R$. They gave relationship between ring-theoretic properties of $R$ and graph-theoretic properties of $\Gamma(R)$. Later, Demeyer, Mckenzie and Schneider in [17] studied graphs on commutative semigroups with 0 in a similar manner.

Also the concept of the zero-divisor graph is studied in posets, meet-semilattices, lattices; see ([32], [39], [40], [22], [29]).

In this chapter we introduce and study the concept of the zero-divisor graph derived from meet-semilattice $L$ with 0 on the lines of Anderson and Livingston [6]. Also we generalize some results from Demeyer, Mckenzie and Schneider [17] to meet-semilattice $L$ with 0. Throughout this chapter $L$ denotes a meet-semilattice with 0.

**Definition 2.1.1.** An element $a \in L$ is called a zero-divisor if there exists a nonzero element $b \in L$ such that $a \land b = 0$.

We denote by $Z(L)$ the set of all zero-divisors of $L$ and $Z^*(L) = Z(L) - \{0\}$. We associate a graph $\Gamma(L)$ to $L$ with vertex set $Z^*(L)$. Two distinct elements $x, y \in Z^*(L)$ are adjacent if and only if $x \land y = 0$ and call this graph as the zero-divisor graph of $L$. We show that $\Gamma(L)$ is always connected with $\text{diam} \Gamma(L) \leq 3$ and we determine when $\Gamma(L)$ is a complete graph or a star graph.

**Definition 2.1.2.** Let $(L, \leq)$ be a meet-semilattice. For any $a, b \in L$ either $a \leq b$ or $b \leq a$ holds then $(L, \leq)$ is called a chain.
2.2 Properties of $\Gamma(L)$

In this section, we show that $\Gamma(L)$ is always connected and has small diameter and girth.

We begin this section with some examples.

*Example 2.2.1.* The zero-divisor graph of a finite meet-semilattice with only one atom is the empty graph. The zero-divisor graph of the meet-semilattice in Figure 2.1 is the empty graph.

However, this does not hold for infinite meet-semilattices with one atom. For consider, the infinite meet-semilattice given in Figure 2.2, where the descending dots represent infinite descending chain. It has only one atom $c$ but its graph $\Gamma(L)$ is an infinite star graph.

![Figure 2.1](image1.png)  
![Figure 2.2](image2.png)
The following example shows that nonisomorphic meet-semilattices may have the same zero-divisor graph.

**Example 2.2.2.** The meet-semilattices in Figures 2.3 and 2.4 are not isomorphic but their zero-divisor graph is the path $P_2$ see Figure 2.5.

![Figure 2.3](image1)

![Figure 2.4](image2)

![Figure 2.5](image3)

**Example 2.2.3.** All connected graphs with less than four vertices can be realized as $\Gamma(L)$ for some meet-semilattice $L$ with 0.

There is only one connected graph $K_2$ with two vertices and one can see that $K_2 = \Gamma(L)$, where $L$ is the meet-semilattice given in Figure 2.6a.

![Figure 2.6](image4)

![Figure 2.6a](image5)

Note that $P_3$ and $K_3$ are the only connected graphs on three vertices. Both are realizable as the zero-divisor graph of a meet-semilattice $L$. 
with 0. The corresponding meet-semilattices are given in Figure 2.3 and Figure 2.7a respectively.

![Figure 2.7](image1)

![Figure 2.7a](image2)

There are eleven graphs with four vertices of which only six are connected. Of these six, the following five graphs shown in Figure 2.8 to Figure 2.12 may be realized as $\Gamma(L)$. The corresponding meet-semilattices are given in Figure 2.8a to Figure 2.12a respectively.

However only Figure 2.8, Figure 2.11 and Figure 2.12 can be realized as the zero-divisor graph of a commutative ring $R$ see [6].

![Figure 2.8](image3)

![Figure 2.8a](image4)

![Figure 2.9](image5)

![Figure 2.9a](image6)
We next observe that the graph $P_4$ cannot be realized as $\Gamma(L)$ for any meet-semilattice $L$ with 0.
Consider the path \( P_4 \) shown in Figure 2.13. Let \( P_4 = \Gamma(L) \) for some meet-semilattice \( L \). Then \( a \land b = b \land c = c \land d = 0 \) and no other meet is zero. If \( a \land d = x \) for some \( x \) then \( x \leq a, x \leq d \) hence \( x \land b = 0 \), \( x \land c = 0 \) that is \( x \) is a common neighbor of \( b \) and \( c \), a contradiction. Hence \( P_4 \) cannot be realized as \( \Gamma(L) \).

**Remark 2.2.1.** We have seen above that \( \Gamma(L) \) can be a 3-cycle or a 4-cycle. But, \( \Gamma(L) \) cannot be an \( n \)-cycle for any \( n \geq 5 \). For, consider an \( n \)-cycle, \( a_1 - a_2 - a_3 - \cdots - a_n - a_1 \) with \( n \geq 5 \). Suppose that \( G = \Gamma(L) \) for some meet-semilattice \( L \) with 0. Let \( a_2 \land a_4 = x \) then \( x \leq a_2, x \leq a_4 \) gives \( x \land a_3 = 0, x \land a_5 = 0 \) that is \( x \) is a common neighbor of \( a_3 \) and \( a_5 \). We note that if \( x = a_4 \) then \( a_4 \leq a_2 \) and hence \( a_4 \land a_1 = 0 \), a contradiction. This shows that \( a_2 \land a_4 \) does not exists. Hence \( \Gamma(L) \) cannot be an \( n \)-cycle for any \( n \geq 5 \).

**Theorem 2.2.1.** A disconnected graph cannot be a graph of any meet-semilattice \( L \) with 0.

**Proof.** Suppose \( G \) is a disconnected graph with components \( G_1 \) and \( G_2 \). Let \( G = \Gamma(L) \) for some meet-semilattice \( L \) with 0. There exist vertices \( x \in G_1 \) and \( y \in G_2 \) such that there is no path between \( x \) and \( y \). Let \( a \in G_1 \) and \( b \in G_2 \) be vertices adjacent to \( x \in G_1 \) and \( y \in G_2 \) respectively. Then \( x \land a = 0, y \land b = 0 \). If \( a \land b = z \) for some \( z \in L \). Then \( x \land z = y \land z = 0 \). Thus \( z \) is a common neighbor of \( x \) and \( y \), a
Let \( L \) be a meet-semilattice with 0. Then \( \Gamma(L) \) is complete if and only if \( x \land y = 0 \) for all \( x, y \in Z^*(L) \). For the meet-semilattice \( \text{L} = M_n = \{0, a_1, \ldots, a_n\} \), where \( a_i \land a_j = 0 \), for all \( i \neq j \), \( \Gamma(L) \) is the complete graph \( K_n \).

We next show that the zero-divisor graphs are all connected and have at most 3 diameter and small girth.

**Theorem 2.2.2.** Let \( L \) be a meet-semilattice with 0, then \( \Gamma(L) \) is connected and \( \text{diam}(\Gamma(L)) \leq 3 \). Moreover, if \( \Gamma(L) \) contains a cycle, then \( \text{gr}\Gamma(L) \leq 4 \).

**Proof.** Let \( x, y \in Z^*(L) \) be distinct. If \( x \land y = 0 \), then \( d(x, y) = 1 \).

So suppose that \( x \land y \neq 0 \), then there are \( a, b \in Z^*(L) - \{x, y\} \) with \( a \land x = b \land y = 0 \).

If \( a = b \) then \( x - a - y \) is a path of length 2; thus \( d(x, y) = 2 \). Thus we may assume that \( a \neq b \). If \( a \land b = 0 \), then \( x - a - b - y \) is a path of length 3 and hence \( d(x, y) \leq 3 \). If \( a \land b \neq 0 \), then \( x - a \land b - y \) is a path of length 2; thus \( d(x, y) = 2 \). Hence \( d(x, y) \leq 3 \) and thus \( \text{diam}(\Gamma(L)) \leq 3 \).

As seen above, there exists a path between any two distinct elements in \( Z^*(L) \) and so \( \Gamma(L) \) is connected.
Now, suppose that $\Gamma(L)$ contains a cycle. If $gr\Gamma(L) \geq 5$, then $\Gamma(L)$ contains an $n$-cycle say $a_1 - a_2 - a_3 - \cdots - a_n - a_1$ with $n \geq 5$.

Let $a_2 \land a_4 = x$, $a_3 \land a_5 = y$, $a_5 \land a_2 = z$ in $L$. Then $x \land y = 0$, $y \land z = 0$, $z \land x = 0$, a contradiction to the assumption that $gr\Gamma(L) \geq 5$. Hence $gr\Gamma(L) \leq 4$.

\[ \square \]

Remark 2.2.2. There exist a lattice $L$ such that $gr\Gamma(L) = 4$, see Figure 2.8 and Figure 2.8a.

Theorem 2.2.3. If $a - x - b$ is a path in $\Gamma(L)$, then either $x$ is an atom in $L$ or $a - x - b$ is contained in a cycle of length $\leq 4$.

Proof. Suppose $a - x - b$ is a path in $\Gamma(L)$ and $x$ is not an atom in $L$ then there is a nonzero $c < x$. Then $a \land c = b \land c = 0$. Hence $a - x - b - c - a$ is a cycle of length equal to 4.

Now we consider meet-semilattices whose graphs contain a cycle.

Theorem 2.2.4. If $L$ does not contain any atom, then any edge in $\Gamma(L)$ is contained in a cycle of length $\leq 4$, and therefore $\Gamma(L)$ is a union of 3-cycles and 4-cycles.

Proof. Let $a - x$ be an edge in $\Gamma(L)$. Since $\Gamma(L)$ is connected and $|\Gamma(L)| \geq 3$, there exists a vertex $b$ in $\Gamma(L)$ with $a - x - b$ or $x - a - b$ is a path in $\Gamma(L)$. 
In the first case, if \( b \land a = 0 \) then \( a - x - b - a \) is a 3-cycle. If \( b \land a \neq 0 \), since \( x \) is not an atom then there exists a nonzero \( c < x \). Then \( a \land c = 0 \), \( b \land c = 0 \). Hence \( a - x - b - c - a \) is a cycle of length 4. Thus \( x \) is contained in a cycle of length \( \leq 4 \), so \( a - x \) is an edge of either a 3-cycle or a 4-cycle. In the second case, if \( x \land b = 0 \) then \( x - a - b - x \) is a 3-cycle. If \( x \land b \neq 0 \), since \( a \) is not an atom then there exists a nonzero \( d < a \). Then \( d \land x = 0 \), \( d \land b = 0 \). Hence \( d - x - a - b - d \) is a cycle of length 4. Thus \( a \) is contained in a cycle of length \( \leq 4 \), so \( a - x \) is an edge of a 4-cycle. Hence \( a - x \) is an edge of a 3-cycle or a 4-cycle.

For any meet-semilattice \( L \), let \( K \), the core of \( \Gamma(L) \), be the union of cycles in \( \Gamma(L) \).

**Theorem 2.2.5.** Let \( L \) be a meet-semilattice with 0. If \( \Gamma(L) \) contains a cycle, then the core \( K \) of \( \Gamma(L) \) is a union of 3-cycles and 4-cycles and any vertex in \( \Gamma(L) \) is either a vertex of the core \( K \) of \( \Gamma(L) \) or is a pendant of \( \Gamma(L) \).

**Proof.** Let \( a_1 \in K \) and suppose that \( a_1 \) does not belong to any 3-cycle or a 4-cycle in \( \Gamma(L) \). Then \( a_1 \) is in some \( n \)-cycle \( a_1 - a_2 - a_3 - \cdots - a_n - a_1 \) with \( n \geq 5 \). By Theorem 2.2.3, \( a_1 \) is an atom in \( L \). Then \( a_1 \leq a_4 \) implies that \( a_1 \land a_3 = 0 \), a contradiction. Hence \( a_1 \) is in a 3-cycle or a 4-cycle.
Now suppose that \( a \) is any vertex in \( \Gamma(L) \). If \( a \not\in K \) and \( a \) is not a pendant vertex then the following possibilities hold. (i) \( a \) is contained in a path of the form \( x - y - a - b \) with \( b \in K \) or (ii) \( a \) is contained in a path of the form \( x - a - b \) with \( b \in K \).

Since \( b \in K \), \( b \) is contained in a 3-cycle or a 4-cycle, say \( b - c - d - b \) or \( b - c - d - e - b \).

In (i), we get \( d(x, c) = 4 \), contradicts \( \text{diam}(\Gamma(L)) \leq 3 \). Hence (i) cannot hold.

In (ii), we get \( x - a - b - c - d - b \) or \( x - a - b - c - d - e - b \). This gives \( a \land c = 0 \), a contradiction as \( a \not\in K \). Thus (ii) cannot hold. Hence either \( a \in K \) or \( a \) is a pendant vertex.

\( \square \)

**Theorem 2.2.6.** Let \( L \) be a meet-semilattice with 0 and \( a \in Z^*(L) \) be a pendant vertex in \( \Gamma(L) \). Let \( x \) be the vertex adjacent to \( a \). Then \( x \) is an atom in \( L \).

**Proof.** If \( \Gamma(L) \) has only two vertices, then the result is trivial. Otherwise suppose \( \Gamma(L) \) has more than two vertices. Since \( a \) is a pendant vertex at \( x \) and \( \Gamma(L) \) is connected, there exists a vertex \( b \) in \( \Gamma(L) \) such that \( x - b \) is an edge in \( \Gamma(L) \). Then \( a - x - b \) is a path in \( \Gamma(L) \) not contained in a cycle, and by Theorem 2.2.3, \( x \) is an atom in \( L \).

\( \square \)

**Theorem 2.2.7.** If \( L \) does not contain any atom, then every pair of
vertices in $\Gamma(L)$ is contained in a cycle of length $\leq 6$.

Proof. Let $a, b$ be vertices of $\Gamma(L)$. If $a - b$ is an edge in $\Gamma(L)$, then by the Theorem 2.2.4, $a - b$ is an edge of a 3-cycle or a 4-cycle.

If $a - b$ is not an edge, then $d(a, b) = 2$ or $d(a, b) = 3$. Suppose $d(a, b) = 2$, then there is a path $a - x - b$ and since $x$ is not an atom, by Theorem 2.2.3, $a - x - b$ is contained in a cycle of length $\leq 4$. If $d(a, b) = 3$, then there exists a path $a - x - y - b$. Since $x, y$ are not atoms, there exist nonzero $c, d \in L$ such that $c < x$ and $d < y$. Then $c \land a = c \land y = 0$ and $d \land b = d \land x = 0$. Thus we get two cycles $a - x - y - c - a$ and $b - y - x - d - b$. Thus there exists a cycle $a - x - d - b - y - c - a$ of length less than or equal to 6 containing vertices $a, b$.

$\square$

Remark 2.2.3. Not every graph having diameter at most 3 and girth $\leq 4$ is the graph of a meet-semilattice. For example, see Figure 2.14.

![Figure 2.14](image)

This is not a graph of a meet-semilattice since $b \land d$ does not exist. If $b \land d = a$, then $a \leq b$, a contradiction since $a$ and $b$ are adjacent.

If $b \land d = b$ then $b \land c = b \land d \land c = 0$, a contradiction since $b$ and $c$ are not adjacent.
If \( b \wedge d = c \) then \( c \leq d \), a contradiction since \( c \) and \( d \) are adjacent.

If \( b \wedge d = d \) then \( d \wedge y = b \wedge d \wedge y = 0 \), a contradiction since \( d \) and \( y \) are not adjacent.

If \( b \wedge d = x \) then \( x \leq d \), a contradiction since \( x \) and \( d \) are adjacent.

If \( b \wedge d = y \) then \( y \leq b \), a contradiction since \( b \) and \( y \) are adjacent.

If \( b \wedge d = z \) for some \( z \in L \) then \( z \leq b, z \leq d \) implies that \( z \wedge y \leq b \wedge y = 0, z \wedge c \leq d \wedge c = 0 \) that is \( z \) is a common neighbor of \( y \) and \( c \), a contradiction. Hence \( b \wedge d \) does not exist.

2.3 Integral Meet-semilattices and Graphs of Product of Meet-semilattices

We say that a meet-semilattice \( L \) with 0 is an integral meet-semilattice if for \( a, b \in L \), \( a \wedge b = 0 \) implies \( a = 0 \) or \( b = 0 \), for example the meet-semilattice in Figure 2.1 is an integral meet-semilattice.

**Theorem 2.3.1.** If \( L_1 \) and \( L_2 \) are integral meet-semilattices with 0 such that \( |L_1| = m + 1 \), \( |L_2| = n + 1 \) and \( L \cong L_1 \times L_2 \), then \( \Gamma(L) \) is the complete bipartite graph \( K_{m,n} \).

**Proof.** If \( L_1 = \{0, a_1, \ldots, a_m\} \) and \( L_2 = \{0, b_1, \ldots, b_n\} \), then the pairs of the form \((a_i, 0)\) and \((0, b_j)\) are all adjacent.

Moreover, no pairs of the form \((a_i, 0), (a_k, 0)\) are adjacent, since
Similarly, no pairs of the form \((0, b_i), (0, b_j)\) are adjacent. The resulting graph is a complete bipartite graph with partitions
\[ A = \{(a_1, 0), \cdots, (a_m, 0)\}, \text{ and } B = \{(0, b_1), \cdots, (0, b_n)\}. \]

\]}

\]}

**Theorem 2.3.2.** Let \( L_1 \) and \( L_2 \) be two meet-semilattices with 0 and \( L = L_1 \times L_2 \). Then \( gr(\Gamma(L)) = \infty \) if and only if either

1. \(|\Gamma(L)| \leq 2\) or
2. \(|\Gamma(L)| = 3\) and \( \Gamma(L) \) is not complete or
3. \( L \cong C_2 \times L_2 \), where \( L_2 \) is an integral meet-semilattice and \( C_2 \) is the two element chain.

**Proof.** Suppose \( gr\Gamma(L) = \infty \), then either \(|\Gamma(L)| \leq 2\) or \(|\Gamma(L)| = 3\) and \( \Gamma(L) \) is not complete. Suppose both these fail.

Case 1: Let \( L_1, L_2 \) be two meet-semilattices but one of these say \( L_1 \) is not an integral meet-semilattice. There exist nonzero elements \( a, b \in L_1 \) with \( a \wedge b = 0 \). Let \( c \in L_2 \) be a nonzero element. Then \((a, 0), (b, 0), (0, c) \in L_1 \times L_2 \) form a 3-cycle in \( \Gamma(L) \), which is a contradiction to \( gr\Gamma(L) = \infty \). Hence both \( L_1, L_2 \) must be integral meet-semilattices.

Case 2: Suppose that both \( L_1 \) and \( L_2 \) are integral meet-semilattices with \(|L_1| > 2, |L_2| > 2\). Then choose nonzero \( a, b \in L_1 \) and \( c, d \in L_2 \). The elements \((a, 0) - (0, c) - (b, 0) - (0, d)\) form a 4-cycle in \( \Gamma(L) \), a
contradiction. Thus either $L_1$ or $L_2$ is $C_2$ and the other is an integral meet-semilattice.

Conversely, suppose either $|\Gamma(L)| \leq 2$ or $|\Gamma(L)| = 3$ and $\Gamma(L)$ is not complete or $L \cong L_1 \times L_2$ where $L_1 = C_2$ and $L_2$ is an integral meet-semilattice. We have to prove that $gr\Gamma(L) = \infty$.

If $|\Gamma(L)| = 2$ then the zero-divisor graph is $P_2$. Hence $gr\Gamma(L) = \infty$.

If $|\Gamma(L)| = 3$ and $\Gamma(L)$ is not complete then the zero-divisor graph is $P_3$. Hence $gr\Gamma(L) = \infty$.

Suppose $L \cong L_1 \times L_2$ where $L_1 = C_2$ and $L_2$ is an integral meet-semilattice. Let $(1, 0) - (0, a_1) - \cdots - (0, a_n)$ be a cycle in $\Gamma(L)$. Then $(0, a_1), (0, a_2)$ are adjacent hence $a_1 \land a_2 = 0$ for nonzero $a_1, a_2 \in L_2$, a contradiction since $L_2$ is an integral meet-semilattice. Hence $gr\Gamma(L) = \infty$. □

**Corollary 2.3.3.** Let $L_1$ and $L_2$ be two meet-semilattices with 0 and $L = L_1 \times L_2$, then $\Gamma(L)$ is star graph if and only if one of the $L_1$ or $L_2$ is $C_2$ and the other is an integral meet-semilattice.

**Proof.** Suppose that $\Gamma(L)$ is a star graph.

Case 1. Let $L = L_1 \times L_2$ and at least one of the $L_1$ or $L_2$ is not an integral meet-semilattice. We may assume that $L_1$ is not an integral meet-semilattice. Then there exist nonzero elements $a, b \in L_1$ with $a \land b = 0$. Choose nonzero $c \in L_2$. Then $(a, 0) - (0, c) - (b, 0)$ form a
3-cycle in $\Gamma(L)$, a contradiction.

Case 2. Suppose $L = L_1 \times L_2$ and both $L_1$ and $L_2$ are integral meet-semilattices with $|L_1| > 2$, $|L_2| > 2$. Choose nonzero $a, b \in L_1$ and $c, d \in L_2$. Then $(a, 0) - (0, c) - (b, 0) - (0, d)$ form a 4-cycle in $\Gamma(L)$, a contradiction. Thus either $L_1$ or $L_2$ is $C_2$ and the other is an integral meet-semilattice.

Conversely suppose $L = L_1 \times L_2$ where $L_1 = C_2$ and $L_2$ is an integral meet-semilattice. Then $(1, 0)$ is adjacent to every other vertex and no other vertices are adjacent to each other that is $\Gamma(L)$ is a star graph. □

**Corollary 2.3.4.** Let $L_1$ and $L_2$ be two meet-semilattices with 0 and $L = L_1 \times L_2$. Then $\Gamma(L)$ does not contain a 3-cycle if and only if $L_1$, $L_2$ are integral meet-semilattices.

**Definition 2.3.1.** Let $L$ be a meet-semilattice with 0. For $x, y \in Z^*(L)$, we define a relation $x \sim y$ if and only if $x \land y = 0$ or $x = y$.

**Corollary 2.3.5.** The relation $\sim$ is transitive (equivalently, an equivalence relation) if and only if $\Gamma(L)$ is complete.

**Proof.** Suppose the relation $\sim$ is transitive but $\Gamma(L)$ is not complete, then there exist $x, y \in Z^*(L)$ with $x \land y \neq 0$. Hence there exist $a, b \in Z^*(L) - \{x, y\}$ with $a \land x = 0 = b \land y$. If $a = b$ then $x - a - y$
is a path and by transitivity $x \land y = 0$. If $a \neq b$ but $a \land b = 0$ then $x - a - b - y$ is a path again by transitivity $x \land y = 0$. If $a \land b \neq 0$ then $x - a \land b - y$ is a path hence by transitivity $x \land y = 0$.

Converse is trivial. \hfill \Box

Remark 2.3.1. For any star graph with $n$ elements there corresponds a meet-semilattice see Figure 2.15.

![Figure 2.15](image)

**Theorem 2.3.6.** Let $L_1$ and $L_2$ be two meet-semilattices with 0 and $L = L_1 \times L_2$. Then exactly one of the following holds:

1. $\Gamma(L)$ has a cycle of length 3 or 4 (that is $\text{gr} \Gamma(L) \leq 4$),

2. $\Gamma(L)$ is a star graph.

**Proof.** Case (1): Suppose $L \cong L_1 \times L_2$, where at least one of $L_1$ and $L_2$ is not an integral meet-semilattice, say $L_1$ is not an integral meet-semilattice. Then there exist nonzero $a, b \in L_1$, with $a \land b = 0$ and choose nonzero $c \in L_2$. Then $(a, 0) - (b, 0) - (0, c)$ form a cycle of length 3 in $\Gamma(L)$. 
Case (2): Let $L \cong L_1 \times L_2$ where $L_1, L_2$ both are integral meet-semilattices with $|L_1| > 2, |L_2| > 2$. Let $a, b \in L_1$, and $c, d \in L_2$ be nonzero elements. Then $(a, 0) - (0, c) - (b, 0) - (0, d)$ form a cycle of length 4 in $\Gamma(L)$.

Case (3): Let $L \cong L_1 \times L_2$, where either $|L_1| = 2$ or $|L_2| = 2$. Let $|L_1| = 2$ and $L_2$ is an integral meet-semilattice then by Theorem 2.3.2, $\Gamma(L)$ is a star graph. 

Nimbhorkar, Wasadikar and Demeyer in [39] have proved the following Theorem.

**Theorem 2.3.7.** Let $L_i, i = 1, 2, \ldots, n$ be lattices with 0 and $\text{clique}(L_i) = m_i$ for $i = 1, 2, \ldots, n$. Let $L = L_1 \times L_2 \times \cdots \times L_n$. Then $\text{clique}(L) = \sum_{i=1}^{n} m_i - n + 1$.

By using above Theorem we get the following Theorem

**Theorem 2.3.8.** Let $L_1, L_2, \ldots, L_n$ be integral meet-semilattices $(n \geq 2)$ and let $L = L_1 \times L_2 \times \cdots \times L_n$. Then $\text{Clique}(\Gamma(L)) = n$.

**Proof.** We have $\text{Clique}(\Gamma(L)) \geq n$. Only it remains to show that $\text{Clique}(\Gamma(L)) \leq n$.

For $n = 2$ it is obvious.

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a maximal clique of $\Gamma(L)$ with $n \geq 3$ and each $x_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$. We may assume that $x_{11} \neq 0$ and
so \( x_{21} = x_{31} = \cdots = x_{m1} = 0 \). Hence we may consider \( X - x_1 \) as a maximal clique of \( \Gamma(L_2 \times L_3 \times \cdots \times L_n) \). Therefore by induction we have \( m - 1 \leq n - 1 \) and so \( m \leq n \). Hence \( \text{Clique}\Gamma(L) = n \).

**Theorem 2.3.9.** Let \( L_1 \) and \( L_2 \) be integral meet-semilattices with \( 0 \) such that \( |L_1| = m + 1 \), \( |L_2| = n + 1 \) and \( L \cong L_1 \times L_2 \), then
\[
diam(\Gamma(L)) \leq 2.
\]

**Proof.** By Theorem 2.3.1, \( \Gamma(L) \) is a complete bipartite graph hence \( \text{diam}(\Gamma(L)) \leq 2 \).

We define circumference of \( G \) as follows.

**Definition 2.3.2.** The circumference of \( G \), denoted by \( c(G) \), is defined as the length of the longest cycle in \( G \).

\[
c(G) = 0 \text{ if } G \text{ contains no cycle.}
\]

\[
c(G) = \infty \text{ if } G \text{ has cycles of arbitrary length. Clearly } gr(G) \leq c(G)
\]

and \( \omega(G) \leq c(G) \) if \( G \) contains a cycle.

**Remark 2.3.2.**
1. \( \omega\Gamma(L) = 2 \) if and only if \( |\Gamma(L)| \geq 2 \) and \( \Gamma(L) \) contains no 3 cycles. Thus \( \omega\Gamma(L) \geq 3 \) if and only if \( \Gamma(L) \) contains 3 cycles.

2. Let \( L = L_1 \times L_2 \) where \( L_1 \) and \( L_2 \) are integral meet-semilattices with \( |L_1| = m \geq 3 \), \( |L_2| = n \geq 3 \). Then by Theorem 2.3.1
\( \Gamma(L) \) is the complete bipartite graph \( K_{m-1,n-1} \) with \( \omega \Gamma(L) = 2 \), \( gr \Gamma(L) = 4 \) and \( c \Gamma(L) = 2\{\min(m,n) - 1\} \).

**Theorem 2.3.10.** Let \( L_1, L_2, \ldots, L_n \) be integral meet-semilattices \((n \geq 2)\) and let \( L = L_1 \times L_2 \times \cdots \times L_n \). Then \( c \Gamma(L) = \infty \) if and only if at least two of the \( L_i \)'s are infinite.

**Proof.** Suppose that \( L_1 \) and \( L_2 \) are infinite meet-semilattices. For each \( m \geq 1 \), we may choose distinct nonzero \( a_1, a_2, \ldots, a_m \in L_1 \) and \( b_1, b_2, \ldots, b_m \in L_2 \). Then \((a_1,0) - (0,b_1) - (a_2,0) - \cdots - (a_m,0) - (0,b_m)\) is a cycle of length \( 2m \). Since \( m \) is arbitrary thus \( c \Gamma(L) = \infty \).

Conversely, we know that if each \( L_i \) is finite then \( c \Gamma(L) < \infty \). Thus we suppose that, \( L_1 \) is infinite and \( L_2, L_3, \ldots, L_n \) are finite.

Let \( x_1 - x_2 - \cdots - x_m - x_1 \) be a cycle of length \( m \) with each \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in L \). Note that, if \( x_{i1} \neq 0 \) then \( x_{i+1} = 0 \). Thus \( m \leq 2|L_2 \times L_3 \times \cdots \times L_n| \).

So \( c \Gamma(L) \) is finite but we have \( c \Gamma(L) = \infty \), a contradiction. Hence at least two \( L_i \)'s are infinite.

Akhtar and Lee in [2] studied the connectivity of the zero-divisor graph of a commutative ring. They studied the vertex and the edge connectivity of the zero-divisor graph \( \Gamma(R) \). They shown that the edge connectivity of \( \Gamma(R) \) always coincides with the minimum degree. Lower and upper bounds are given for the vertex connectivity.
Definition 2.3.3. A separating set or *vertex cut* of a graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one component.

Definition 2.3.4. The connectivity of $G$, written as $k(G)$, is the minimum size of a vertex set $S$ such that $G - S$ is disconnected or has only one vertex.

Definition 2.3.5. A disconnecting set of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component.

Definition 2.3.6. The edge connectivity of $G$, denoted by $\lambda(G)$, is the minimum size of a disconnecting set.

Definition 2.3.7. The minimal degree $\delta(G)$ of $G$, is defined as $\min \{\deg(v) \mid v \in V\}$.

The following well known result can be found in any standard textbook on graph theory; see for example ([19], [31], [53]).

**Theorem 2.3.11.** If $G$ is a simple graph, then $k(G) \leq \lambda(G) \leq \delta(G)$.

**Proposition 2.3.12.** Let $L_1$ and $L_2$ be integral meet-semilattices. Then $k\Gamma(L_1 \times L_2) = \delta\Gamma(L_1 \times L_2) = \min(|L_1^*|, |L_2^*|)$.

**Proof.** The nonzero zero-divisors of $L_1 \times L_2$ are either of the form $(a, 0)$, $a \in L_1^*$ or $(0, b)$, $b \in L_2^*$. Since $L_1$ and $L_2$ are integral meet-semilattices we get $\Gamma(L_1 \times L_2)$ is a complete bipartite graph $K_{m,n}$ with $m = |L_1^*|$ and $n = |L_2^*|$.
and $n = |L_2^*|$. Then by Theorem 2.3.11 $k\Gamma(L_1 \times L_2) = \delta\Gamma(L_1 \times L_2)$ and it is well know that the connectivity of such a graph is $\min(m, n)$.

Thus $k\Gamma(L_1 \times L_2) = \delta\Gamma(L_1 \times L_2) = \min(|L_1^*|, |L_2^*|)$. \qed